

Lecture 25: More Uniform Convergence

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Recall: Suppose $A \subset \mathbb{R}$, $f: A \rightarrow \mathbb{R}$ bounded. The sup norm of f is $\|f\|_\infty = \|f\| := \sup_{a \in A} |f(a)|$

\rightarrow A sequence f_n of functions on A converges uniformly to f if $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$

Recall: Uniform convergence \implies pointwise convergence.

non-eg: $A = (0, 1)$ $f_n(x) = x^n$; if this converges uniformly its (uniform) limit must be its pointwise limit.

\rightarrow If I fix $a \in (0, 1)$, then I do know $\lim_{n \rightarrow \infty} a^n = 0$

\rightarrow If $\{f_n\}$ converges uniformly, its limit must be the zero function. Uniform convergence asks about

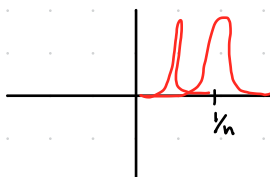
$\|x^n\|$ computed on domain $A = (0, 1) \rightarrow \|x^n\| := \sup_{a \in (0, 1)} a^n = 1$ $\lim_{x \rightarrow 1} x^n = 1$ $\lim_{n \rightarrow \infty} 1 \neq 0$ so no uniform convergence.

Example: $A = (0, 0.25)$ $f_n(x) = x^n$ $\|x^n\|_{on A} = \frac{1}{4^n} \rightarrow 0$ as $n \rightarrow \infty$ so on A , the sequence does converge uniformly.

Theorem: If all the functions in the sequence f_n are continuous on A and $f_n \rightarrow f$ uniformly then f is continuous.

More generally: If $f_n \rightarrow f$ uniformly and $\forall n, \lim_{x \rightarrow c} f_n(x)$ exists, then $\lim_{x \rightarrow c} f(x) = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow c} f_n(x) \right)$

Example: $A = \mathbb{R}$ $f_n(x) = e^{(-n^4(x-1/2)^2)}$



this is an example that showcases that the above theorem is not an iff ∇ the example $f_n(x) = x^n, c=1, A=[0, 1]$ shows that (*) can fail if you drop uniformity.

Example: $A = (0, 1)$ $f_n(x) = x^n(1-x)$ does converge uniformly to 0.

Theorem re-written: Let $A \subset \mathbb{R}$ and c a cluster point of A (possibly $c \in A$). Suppose $f_n: A \rightarrow \mathbb{R}$ is a sequence of functions s.t.

- $\forall n, \lim_{x \rightarrow c} f_n(x)$ exists
- $f_n \rightarrow f$ uniformly

then $\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x)$ \star

Recall: Cauchy Criterion ①. if x_n is a sequence of numbers, then $\lim_{n \rightarrow \infty} x_n$ exists $\iff \forall \epsilon > 0, \exists N(\epsilon)$ s.t. if $n, m > N(\epsilon)$ then $|x_n - x_m| < \epsilon$

②. $\lim_{x \rightarrow c} f(x)$ exists $\iff \forall \epsilon > 0, \exists \delta(\epsilon)$ s.t. if $x, y \in V_{\delta(\epsilon)}(c)$ then $|f(x) - f(y)| < \epsilon$

③. A sequence of functions f_n converges uniformly $\iff \forall \epsilon > 0, \exists N$ s.t. $\|f_m - f_n\| < \epsilon, \forall m, n > N$.

WTS: $\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x)$ exists Equiv: WTS: "if x, y are both close to c then $|\lim_{n \rightarrow \infty} f_n(x) - \lim_{n \rightarrow \infty} f_n(y)|$ is small."

Idea: if $x, y \approx c$... will finish next week

$$\lim_{n \rightarrow \infty} (f_n(x) - f_n(y))$$