Lecture 26: Interchange of limits

30·Mar·2026

"Interchange of limits theorem"

Theorem: Fix $A \subset \mathbb{R}$, $f_n : A \to \mathbb{R}$ a sequence of functions. Suppose $c \in A$ s.t. every f_n is continuous at c. Suppose f_n converges uniformly on A. Then $\lim_{n\to\infty} f_n = f$ is continuous at c.

stupid case: c is an isolated point in A. Then every function $f : A \to \mathbb{R}$ is contin. at c. So nothing to prove.

Interesting case: c is not an isolated point. Then "f_n is cont. at c" is $\lim_{x\to c} f_n(x) = f_n(c)$. Statement of the theorem asserts $\lim_{x\to c} f(x) = f(c)$

→ Theorem is asserting that $\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$ ⚠ This very much requires uniform convergence

Proof: uniform convergence is the statement that $\forall n \geq N(\varepsilon), \forall x, |f_n(x) - f(x)| < \varepsilon$. So in particular $|f_N(x) - f(x)| < \varepsilon/3 \ \forall x \in A$

Goal: control $|f(x) - f(c)|$ when $x \approx c$ → $|f(x) - f(c)| \leq \underbrace{|f(x) - f_N(x)|}_{<\varepsilon/3} + |f_N(x) - f_N(c)| + \underbrace{|f_N(c) - f(c)|}_{<\varepsilon/3}$ where $N = N(\varepsilon)$

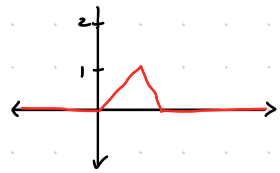
$\because f_N$ is cont. at c, $\exists \delta(\varepsilon/3, N(\varepsilon/3))$ s.t. $|f_N(x) - f_N(c)| < \varepsilon$ whenever $x \in V_\delta(c)$ so $|f(x) - f(c)| < \varepsilon$ if $x \in V_\delta(c)$

→ In particular, if f_n are all cont. everywhere and $f_n \to f$ uniformly, then f is continuous everywhere.

Not iff statement.

Example: $A \subset \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x \leq 0 \\ nx & 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases}$$



$$\lim_{n\to\infty} f(x) = \begin{cases} 0 & x \leq 0 \\ 0 & x > 0 \end{cases} = 0$$

not uniform: $\|f_n - 0\| = 1 \ \forall n$

Theorem (Dini): if $A = [a,b]$ a closed interval and f_n is monotonic (decreasing) i.e. $f_n(x) \geq f_{n+1}(x) \ \forall x \in A$; $\forall n$ and all f_n and $f := \lim_{n\to\infty} f_n$ are all continuous then the convergence is uniform.

Closed intervals are special: Given $A \subset \mathbb{R}$, a covering of A is some set of open intervals $\{(a_i, b_i)\}_{i\in I}$ (could be infinite) s.t. $A \subset \bigcup_{i\in I} (a_i, b_i)$

Example: $A = (0,1)$ $\left\{(0, \frac{1}{2}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{4}, \frac{3}{4}), (\frac{3}{5}, \frac{4}{5}), \dots, (\frac{n-1}{n+1}, \frac{n}{n+1})\right\}_{n=1,2,\dots}$ this does cover $(0,1)$ and it takes infinitely many unions.

Theorem: If $A = [a,b]$ is a closed interval then for any covering $\{(a_i, b_i)\}_{i\in I}$ this is a finite subset $J \subset I$ s.t. such that A is covered by $\{(a_i, b_i)\}_{i\in J}$

Proof: Lets define an equivalence relation in A as follows: $x \sim y$ if $\exists i$ s.t. $x, y \in (a_i, b_i)$ → right now not transitive

→ More generally $x \sim y$ if there is a finite sequence $x \sim z_1 \sim z_2 \sim \cdots \sim z_n \sim y$ where each consecutive pair are in the same interval (a_i, b_i)

WTS: $a \sim b$ $\{c \mid a \sim c\} = E \subset A = [a,b]$

- This set is not empty $a \in E$
- Supremum of $E \in A$ $e := \sup E$ so $\exists i$ s.t. $e \in (a_i, b_i)$
  so $\exists \varepsilon > 0$ s.t. $e - \varepsilon \in (a_i, b_i) \Longrightarrow e - \varepsilon \in E$ so $e \in E$
  if $e \neq b$ then $e + \varepsilon \in E$ contradiction.