Lecture 27: Dini and Profound Truths

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Last time: Given $A \subset \mathbb{R}$, a covering of A by open intervals is a set of open intervals $\{U_i\}_{i\in I}$ $U_i = (a_i, b_i)$ s.t. $\bigcup_{i\in I} U_i \supseteq A$ $\to$ A subcovering of $\{U_i\}_{i\in I}$ is a subset of $\{U_{i\in I}\}$ which is also a covering.

Theorem: If $A = [a,b]$ is a closed interval, then for any covering of A by open intervals you can already cover A by some finitely many opens in your covering. $\to$ "Any covering of $[a,b]$ admits a finite subcovering"

Theorem (Dini): Suppose $A=[a,b]$ is a closed interval and $\{f_n\}_{n\in\mathbb{N}}$ are a sequence of continuous functions, and $\forall x \in A$ $\lim_{n\to\infty} f_n(x)$ exists and $f(x) := \lim_{n\to\infty} f_n(x)$ is continuous and $\forall x$, $f_n(x) \geq f_{n+1}(x)$ "$f_n \geq f_{n+1}$"
Then the convergence $f_n \to f$ is uniform.

Proof: Set $g_n := f_n - f$ $\to$ still continuous, still monotone. *We know that $g_n \to 0$ pointwise. WTS: $g_n \to 0$ uniformly

* Know that $\forall x$: $\forall \varepsilon > 0$, $\exists N(x,\varepsilon)$ s.t. $g_N(x) < \varepsilon$.

** Know that each g_n is continuous everywhere $\to$ look at what continuity of $g_{N(x,\varepsilon)}$ means at x

$\to$ $\exists \delta(x,\varepsilon,N)$ s.t. if $y \in V_\delta(x)$ then $|g_N(y) - g_N(x)| < \varepsilon$

$\to$ In particular $\forall \varepsilon>0, \forall x, \exists N(\varepsilon,x), \exists \delta(\varepsilon,x)$ s.t. $g_N(y) < 2\varepsilon$ if $y \in V_\delta(x)$

$\to$ This collection $V_{\delta(\varepsilon,x)}(x)$'s is a covering of $[a,b]$ $\to$ Find a finite subcovering

A finite subcover of this covering is

- a finite list $x_1, x_2, \dots, x_K$ with K some finite #
- for each x_i we have a δ_i
- for each x_i we have a N_i s.t.   Set $M := \max(N_1, N_2, \dots, N_K)$
  - ** if $y \in V_{\delta_i}(x_i)$ then $g_{N_i}(y) < 2\varepsilon$ $\to$ if $y \in V_{\delta_i}(x_i)$ then $g_M(y) < 2\varepsilon$
  - ** These finitely many $V_{\delta_i}(x_i)$'s cover $[a,b]$ $\to$ so if $y \in [a,b]$ then $g_M(y) < 2\varepsilon$ $\|g_M\| < 2\varepsilon$

Summarize: $\forall \varepsilon > 0, \exists M$ s.t. $\dots \leq \|g_{M+1}\| \leq \|g_M\| < \varepsilon$ so $g_M \to 0$ converges uniformly.

Definition: a set $U \subset \mathbb{R}$ is open if $\forall x \in U, \exists \delta$ s.t. $V_\delta(x) \subseteq U$

Example: any open interval $(\quad (\,|\,))$ **non-e.g.:** A closed interval $[\quad (\,|\,]\,)$

Profound truths: 1). Any (possibly infinite) union of opens is open
2). Every open is a (possibly infinite) union of open intervals.

Definition: $C \subseteq \mathbb{R}$ is closed if $C' = \mathbb{R} \setminus C$ is open. (C': Complement)

Theorem: For a subset $C \subset \mathbb{R}$, TFAE:

a). C is closed
b). C contains all of its cluster points
c). For any sequence x_n in C, if $\lim x_n$ exists (in $\mathbb{R}$), then the limit is in C.

Proof: $(a \Rightarrow b)$ Given $c \in cl(C)$ wts: $c \in C$, we know that C' is open
$\forall$ small nbhd of c intersects C so c cannot be in C'. Indeed if $c \in C'$ then since C' is open $\exists V_\delta \ni c$ fully inside C'

$(b \Rightarrow a)$ WTS: C' open. Know C contains its cluster pts. Given $c \in C'$, it cannot be a cluster point (by assumption). That means $\exists V_\delta \ni c$ which does not intersect C i.e. $V_\delta \subseteq C'$

Examples:
- $\mathbb{R}$ is open
- $\mathbb{R}$ is closed
- $\emptyset$ is open
- $\emptyset$ is closed

fact: these are the only clopen subsets of $\mathbb{R}$

- $\mathbb{Q}$ is neither closed or open