Lecture 28: Topology

06.Apr.2026

Definition: * A subset $U\subseteq\mathbb{R}$ is open if $\forall x\in U\ \exists\varepsilon>0$ s.t. $V_\varepsilon(x)\subseteq U$

* A subset $C\subseteq\mathbb{R}$ is closed if C' is open

Proposition: C is closed $\Longleftrightarrow$ C contains all of its cluster points.

Proposition: Any union (even infinite union) of open sets is open.

Corollary: Any intersection (even infinite intersection) of closed sets is closed ! $\bigcap^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)=\{0\}$ is not open

Proposition: Any finite intersection of opens is open.

Proof 1: By induction it suffices to show that if U,V are open then $U\cap V$ is open.

$$U=\bigcup_{i\in I}\underbrace{(a_i,b_i)}_{U_i}\qquad V=\bigcup_{j\in I}\underbrace{(c_i,d_i)}_{V_j}\Longrightarrow U\cap V=\bigcup_{i,j\in I}(U_i,V_j)$$

But the intersection of two open intervals is again open.

Proof 2: Given $U=\bigcap_{i=1}^{n}U_i$ is a finite union of opens and given $x\in U$, since each U_i is open, $\exists\varepsilon>0$ s.t. $V_{\varepsilon_i}(x)\subseteq U_i$ Set $\varepsilon:=\min_{i=1}^{n}(\varepsilon_i)$ $\forall\varepsilon>0$, $V_\varepsilon(x)\subseteq V_{\varepsilon_i}(x)\subseteq U_i$ so $V_\varepsilon(x)\subseteq U$

Theorem: Every open subset $U\subseteq\mathbb{R}$ is a disjoint union of countably many open intervals.

Proof: Given $x\in U$ look at $a_x:=\begin{cases}\inf\{a\mid[a,x]\subseteq U\} & \text{if exists}\\ \text{or } -\infty & \text{else}\end{cases}$ $\quad b_x:=\begin{cases}\sup\{b\mid[x,b]\subseteq U\}\\ +\infty\end{cases}$

- $a_x\notin U$ because then $a_x-\varepsilon\in U$ for small enough ε
- $a_x+\varepsilon\in U$ for small enough ε

$(a_x,x]\subseteq U$ and $a_x\notin U$ → then for the same reason $[x,b_x)\subseteq U$ $b_x\notin U$

→ Given x, I have found an interval $I_x:=(a_x,b_x)=\bigcup_{\substack{a_x<a<x\\ x<b<b_x}}[a,b]$

if $y\in I_x$ then $I_y=I_x$

if $y\notin I_x$ then $I_y\cap I_x=\emptyset$

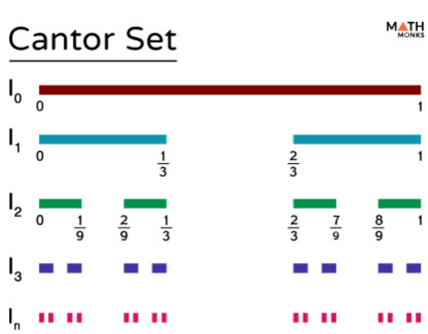
→ $\pi_0U:=U/_{x\sim y}$ if $I_x=I_y$ then $U=\bigcup_{[x]\in\pi_0U}I_x$

→ Every (a,b) contains some rational number so at most $|\mathbb{Q}|$ many I's.

Cantor's Set: → take this infinite intersection C → C is closed

→ $0\in C$ and $\frac{1}{3}\in C$, ...

→ If $a,b\in C$ and $a\neq b$ then $\exists\ a<c<b$ s.t. $c\notin C$ → Completely disconnected



In base 10 $0.999\ldots=1.00\ldots$

In base 3 $0.222\ldots=1.000\ldots$

→ $C=$ The real numbers in $[0,1]$ with a base 3 expansion using only 0's and 2's

→ $|C|=|2^{\mathbb{N}}|=|P(\mathbb{N})|$ → uncountable

→ every point in C is a cluster point.

Definition: Given two subsets $A,B\subseteq\mathbb{R}$, a homeomorphism $A\simeq B$ is a continuous function $f:A\to B$ which is a bijection s.t. $f^{-1}:B\to A$ is also continuous.

E.g.: $\{r \in [0,1] \mid$ base 5 expansion uses only $0,2,4\}$ is homeomorphic to the Cantor set C.

Cantor Set Complement: C' is some countable union.

→ Given $A \subset \mathbb{R}$, $\{a \in A \mid \exists \varepsilon > 0 \text{ s.t. } V_\varepsilon(a) \subseteq A\} = \mathring{A}$ $\qquad \mathring{C} \neq \emptyset$
int(A)
→ $\mathring{A}$ is the maximal open set inside A
→ A is open $\Longleftrightarrow \mathring{A} = A$

→ closure$(A) = \bar{A} = A \cup \text{cl}(A)$ intersection of all closed $C \supseteq A$ $\quad C$ is closed $\Longleftrightarrow \bar{C} = C$
$\mathring{A} \subset A \subset \bar{A}$ "boundary of A"
$\bar{\mathbb{Q}} = \mathbb{R} \qquad \mathring{\mathbb{Q}} = \emptyset$