Lecture 29:

08.Apr.2028

Recall: Let $A\subset\mathbb{R}$ be a covering aka open cover of A, is a set of open subsets $U_i\subset\mathbb{R}$ $\{U_i\}_{i\in I}$ s.t. the union of all of them contains $A \longrightarrow A\subseteq \bigcup_{i\in I} U_i$

$\rightarrow$ A subcover of $\{U_i\}_{i\in I}$ is a subset that still covers A.

E.g.: $A=\mathbb{R}$ have $U_r=(r,r+1)$ $r\in\mathbb{R}$ hence $\{(r,r+1)\}_{r\in\mathbb{R}}$ covers $\mathbb{R}$ and a subcover would be $\{(r,r+1)\}_{r\in\mathbb{Q}}\subseteq\{(r,r+1)\}_{r\in\mathbb{R}}$

Definition: $K\subset\mathbb{R}$ is compact if $\forall$ open cover of K, $\exists$ a finite subcover.

E.g.: During the proof of Dini's theorem, we showed: closed intervals are compact.

non-eg: $\mathbb{R}$ is not compact. $\rightarrow$ $\{(r,r+1)\}_{r\in\mathbb{R}}$ does not admit a finite subcover.

Lemma: Any unbounded set is not compact.

proof: since $\{(r,r+1)\}_{r\in\mathbb{R}}$ covers $\mathbb{R}$ it covers any subset of $\mathbb{R}$

$\rightarrow$ For any finite subset of $\{(r,r+1)\}_{r\in\mathbb{R}}$, the union of that subset is a finite union of bounded sets, hence bounded.

$\rightarrow$ So cannot cover an unbounded set.

Proposition: if $A\subset\mathbb{R}$ is not closed then A is not compact.

Proof: Pick $c\in cl(A)\setminus A$ $\quad \star=\{(-\infty,c-\varepsilon)\cup(c+\varepsilon,\infty)\}_{\varepsilon\in\mathbb{R}_{>0}}$ $\quad \bigcup_{\varepsilon\in\mathbb{R}_{>0}}\{(-\infty,c-\varepsilon)\cup(c+\varepsilon,\infty)\}=\mathbb{R}\setminus c\supset A$

So $\star$ is a cover of A (because $c\notin A$). Because $c\in cl(A)$, can find sequence a_n in A converging to c.

$\rightarrow$ If you only give yourself finitely many ε's, then $\bigcup_{\text{finitely many}}(-\infty,c-\varepsilon)\cup(c+\varepsilon,\infty)$ misses some actual interval around c.

$\rightarrow$ The sequence a_n eventually enters the "missed" interval.

$\rightarrow$ So this finite union is not a cover of A.

Heine-Borel Theorem: $K\subset\mathbb{R}$ is compact $\Longleftrightarrow$ K is closed and bounded.

proof: WTS: if A is bounded but not compact, then A cannot be closed.

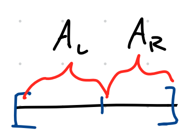
i.e. $\star$ $A\subseteq$ of some closed I (interval) $\quad$ I will use I

$\star$ A has some open cover $\{U_y\}_{y\in Y}$ $\quad$ I will use cover $\rightarrow = G$

Want to construct some $c\in cl(A)\setminus A$

Divide $I=I_L\cup I_R$, $I=[a,b]$, $I_L=[a,\frac{a+b}{2}]$, $I_R=[\frac{a+b}{2},b]$

$A^{(0)}=A\subseteq I$ $\quad A_L:=A\cap I_L$ $\quad A_R:=A\cap I_R$



$\rightarrow$ G is a cover of each of A_L and A_R

$\rightarrow$ For at least of of A_L or A_R, G does not admit a finite subcover

$\rightarrow$ if A_L were covered by finitely many entries in G and also A_R were covered by finitely many entries in G, then $A_L\cup A_R$ would be covered by (finite $\cup$ finite)ly many $g\in G$

Set $A^{(1)} \subseteq I^{(1)}$ be whichever half needed infinitely many $g \in G$. Repeat $A^{(2)} \subseteq I^{(2)}$
$A^{(3)} \subseteq I^{(3)}$
$\vdots$

→ we have built: A nested sequence of closed intervals

$I = I^{(0)} \supseteq I^{(1)} \supseteq I^{(2)} \supseteq \cdots$ where length of $I^{(n)}$ is $\frac{1}{2^n}$ length of I

Set c to be the unique point in $\bigcap_{n=0}^{\infty} I^{(n)}$ $\quad I^{(n)} := [a^{(n)}, b^{(n)}]$

→ $a^{(n)}$ is monotone increasing and bounded $c = \lim a^{(n)}$ exists

→ $b^{(n)}$ is monotone dec and bounded $c = \lim b^{(n)}$ exists

$a^{(n)} \leq c \leq b^{(n)}$ so $c \in \bigcap_{n=0}^{\infty} I^{(n)}$

→ $\dfrac{\text{length } I^{(n)}}{2^n} = |b^{(n)} - a^{(n)}| \to 0$

WTS: $c \in cl(A)$ $\quad c \notin A$

Proof that $c \in cl(A)$: it took infinitely many $g \in G$ to cover $A^{(n)} \subseteq A$ so $A^{(n)} \neq \emptyset$
pick any δ for large enough n, $A^{(n)} = I^{(n)} \subseteq V_\delta(c)$
so $A \cap V_\delta(c) \neq \emptyset$ so $c \in cl(A)$

→ If $c \in A$, then $\exists g \in G$ s.t. $c \in g$

→ This g is open, so $\exists \delta$ s.t. $V_\delta(c) \subseteq g$

→ So for any large enough n, $A^{(n)} \subseteq g$

→ But we chose $A^{(n)}$ so that it could be covered by finitely many g's. a contradiction.

Corollary: if K is compact and C is closed then $K \cap C$ is compact
e.g. Cantor set is compact.