# Lecture 30:

09.Apr.2026

**Theorem:** Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Then $f$ is continuous $\iff$ $f^{-1}(U)$ is open whenever $U \subseteq \mathbb{R}$ is open.

**Comparison:**
- $V, W$ vector spaces and $f: V \to W$ linear if it preserves addition & scalar multiplication
  $f(v_1 + v_2) = f(v_1) + f(v_2) \quad f(\lambda v_1) = \lambda f(v_1)$
- $G, H$ groups $f: G \to H$ is a homomorphism if preserves multiplication $f(gg') = f(g)f(g')$

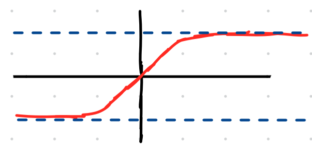
! not true that $f$ is continous $\iff$ $\underbrace{f(U)}_{\{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in U\}}$ is open whenever $U$ is open. **e.g.:** $\sin: \mathbb{R} \to \mathbb{R}$ is continuous.

**proof:** ($\Rightarrow$) suppose $f$ is continuous. Given $U \subseteq \mathbb{R}$ open and $x \in f^{-1}(U)$ then $\underbrace{f(x) \in U}_{\text{meaning of } "f^{-1}(U)"}$; So $\exists \varepsilon > 0$ s.t. $\underbrace{V_\varepsilon(f(x)) \subset U}_{\text{meaning "} U \text{ is open"}}$

Since $f$ is continuous, $\exists \delta$ s.t. $f(V_\delta(x)) \subset \overset{\in U}{V_\varepsilon(f(x))}$; So $V_\delta(x) \subset f^{-1}(U)$.

($\Leftarrow$) $V_\varepsilon(f(x))$ is open, so $f^{-1}(V_\varepsilon(f(x)))$ is open and $x \in \underbrace{f^{-1}(V_\varepsilon(f(x)))}_{U'}$; so $\exists \delta$ s.t. $V_\delta(x) \subset U'$

**Example:** $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \to$ So does not take closed sets to closed sets.



**Theorem:** If $K \subset \mathbb{R}$ is compact and $f: \mathbb{R} \to \mathbb{R}$ is continuous then $f(K)$ is compact.

**Proof:** let $K$ be compact and $f$ continuous. Consider an open cover $\mathcal{G} = \{U_i\}_{i \in I}$ of $f(K)$ i.e. $\bigcup_{i \in I} U_i \supseteq f(K)$
i.e. $\forall k \in K\ \exists i$ s.t. $f(k) \in U_i$ then $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover $K$.
i.e. $\bigcup f^{-1}(U_i) \supseteq K$ i.e. $\forall u \in K, \exists i$ s.t. $f(k) \in U$
$\to$ So $\exists$ a finite subcover $\{f^{-1}(U_i)\}_{i \in \text{some finite subset of } I}$ i.e. $K \subset \bigcup_{i \in \text{finitely many}} f^{-1}(U_i)$

$\to$ So $\{U_i\}_{i \in \text{finite subset}}$ is a cover of $K$. **Recall:** If $f$ is continous and $x_n \to x$ then $f(x_n) \to f(x)$

**Theorem:** $x_n \to x \iff \forall U \ni x$ open all but finitely many $x_n$'s are in $U$.

$\to$ Really, to talk about notions like
- i) • Continuous functions
- ii) • Convergent Sequence

all I really need is some ambient set $R$; some collection of subsets $U \subset R$ named "open" s.t. any finite intersection of opens is opens, and any union of opens is open. $\to$ "A Topology"

**E.g.:** $R$ = all functions from $\mathbb{R}$ to $\mathbb{R}$

**Recall:** in $\mathbb{R}$, defined open $\equiv$ union of open intervals.
i.e. in $\mathbb{R}$, a basic open is an open interval and open $\equiv$ union of basic opens. $\to$

**Definition:** a basis for a topology is a set of basic opens s.t. open $\equiv$ union of basics. Given a proposed collectio of basics "open $\equiv$ union of basics"; automatically get satisfies ii). Get i) if pointwise intersetions of basics are union of basics.

**Pointwise topology:** **Declare:** for any $a(x), b(x): \mathbb{R} \to \mathbb{R}$ $\quad \{f(x) \mid a(x) < f(x) < b(x)\} = \text{"}(a,b)\text{"}$ → for this declaration

intersection of basics is basic.

→ $(a(x), b(x)) \cap (c(x), d(x))$
$= (\max(a(x), c(x)), \max(b(x), d(x)))$

**Uniform topology:** **Declare:** for any function $a(x): \mathbb{R} \to \mathbb{R}$ and any number $r \in \mathbb{R}$, $\{f(x) \mid a(x) < f(x) < a(x) + r\}$

**Example:** $\mathbb{R}^n$ or some geometric object (i.e. some notion of "distanc" with triangle inequality)

→ declare $\forall x \in X$, $\forall \varepsilon > 0$ $V_\varepsilon(x) := \{y \mid \text{distance}(x,y) < \varepsilon\}$ are basic.