MATH 4057/5057: Lie Theory

Assignment 1

suggested due date: January 30

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. (a) Show that $\pi_0 O_n$ has two components.
   (b) Show that the centre of O_n is $\{\pm I\}$, where I denotes the identity matrix.
   (c) Show that if n is odd then the centre of SO_n is trivial and $O_n \cong SO_n \times \{\pm I\}$.
   (d) Show that if $n \geq 4$ is even then the centre of SO_n is $\{\pm I\}$ and O_n is a semidirect product $SO_n \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts by a nontrivial outer automorphism of SO_n.
2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Construct an isomorphism of $GL_n(\mathbb{K})$ with a closed subgroup of $SL_{n+1}(\mathbb{K})$.
3. (a) Show that the map $\mathbb{C}^\times \times SL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ given by $(z, g) \mapsto zg$ is a surjective homomorphism of Lie groups and find its kernel.
   (b) What about the case $\mathbb{R}^\times \times SL_n(\mathbb{R}) \to GL_n(\mathbb{R})$?
4. Recall that $U(1) = \{z \in \mathbb{C} : \|z\| = 1\}$. Given $(a, b) \in \mathbb{R}^2 \smallsetminus \{(0, 0)\}$, define $R_{a,b}$ to be the image of the map $\mathbb{R} \to U(1)^2$ given by $t \mapsto (e^{2\pi i a t}, e^{2\pi i b t})$.
   (a) Suppose that $\frac{a}{b} \in \mathbb{Q}P^1 = \mathbb{Q} \cup \{\frac{1}{0}\}$; write it in lowest terms as $\frac{a}{b} = \frac{m}{n}$, $m, n \in \mathbb{Z}$. Show that $R_{a,b}$ is a closed subgroup of $U(1)^2$ which wraps around the first $U(1)$ component exactly m times and around the second component exactly n times.
   (b) Suppose that $\frac{a}{b} \notin \mathbb{Q}$. Show that $R_{a,b}$ is dense in $U(1)^2$, and is a Lie subgroup diffeomorphic to $\mathbb{R}$ and is not a regular Lie subgroup.
5. A *large manifold* is the same as a manifold except the second-countability condition is dropped — a space locally homeomorphism to $\mathbb{R}^n$, with smooth transition functions. A *large Lie group* is a group object in large manifolds. Show that connected large Lie groups are not large. In other words, show that a connected Lie group, without assuming second-countability, is always necessarily second-countable.
6. Check carefully that if G is a Lie group, and $G^{sc} \to G$ is its simply connected cover, then G^{sc} is also Lie. We explained in lecture why it is a topological group — the main question is to explain why it is a manifold.

7. (a) Show that every algebra automorphism of $\mathbb{H}$ is inner. In other words, show that if $g : \mathbb{H} \to \mathbb{H}$ is an automorphism, then there exists $h \in \mathbb{H}^\times$ such that g is the conjugation by h. What is the space of all such h's?

   (b) Decompose $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$ into real and imaginary parts. Given $g \in SO_3$, consider the action of g on $\mathbb{H}$ that fixes the real part and rotates the imaginary part via g. Show that this action is an automorphism, thereby producing a homomorphism $SO_3 \to \mathrm{Aut}(\mathbb{H})$. Show that this homomorphism is an isomorphism.

   (c) Why does (7a) thus imply that SO_3 is diffeomorphic to $\mathbb{R}P^3$?

8. We explained in class a 2-to-1 group homomorphism $SU_2 \to SO_3$. There is also an obvious 2-to-1 group homomorphism $U_1 \to SO_2$ (find it!). And there is a particularly obvious 2-to-1 group homomorphism $O_1 \to SO_1$ (what is SO_1?).

   (a) Find a 2-to-1 group homomorphism $SU_2 \times SU_2 \to SO_4$. Hint: $2 \times 2 = 4$. Alternate hint: $SU_2 = Sp_1 \subset \mathbb{H}$ can act on $\mathbb{H}$ from either side.

   (b) ! Find a 2-to-1 group homomorphism $Sp_2 \to SO_5$. Hint: $\binom{4}{2} - 1 = 5$.

   (c) ! Find a 2-to-1 group homomorphism $SU_4 \to SO_6$. Hint: $\binom{4}{2} = 6$.

   (d) !! The pattern $\mathbb{R}, \mathbb{C}, \mathbb{H}$ can be continued in various ways. One of them, due to Cayley and Dickson, is as follows. As we described in class, each of these is a $*$-algebra. Given a $*$-algebra A, define

$$\mathrm{CD}(A) := A \oplus At, \qquad (a_1 + b_1 t) \cdot (a_2 + b_2 t) = (a_1 a_2 - b_1 b_2^*) + (a_1 b_2 + b_1 a_2^*)t.$$

   Show that $\mathrm{CD}(A)$ is commutative iff $a^* = a$, and that $\mathrm{CD}(A)$ is associative iff A is commutative. Conclude that $\mathrm{CD}(\mathbb{H})$ is neither associative nor commutative.

   The nonassociative algebra $\mathbb{O}$ is called the *octonions*. Define the *associator* to be the multilinear functional $\alpha : \mathbb{O}^{\otimes 4} \to \mathbb{R}$ given by

$$\alpha(x, y, z, w) := \langle (xy)z - x(yz), w \rangle$$

   where $\langle x, y \rangle = \mathrm{Re}(xy*)$ denotes the standard dot product on $\mathbb{O} = \mathbb{R}^8$. Show that α is alternating.

   Find a 2-to-1 group homomorphism $\{g \in SO_8 : \alpha(gx, gy, gz, gw) = \alpha(x, y, z, w)\} \to SO_7$.

These groups $O_1, U_1, SU_2, SU_2 \times SU_2, Sp_2, SU_4$, and the group from (8d) are the first few *Spin* groups. For each n, there is a specific group $Spin_n$ with a 2-to-1 map $Spin_n \to SO_n$; when $n \geq 3$, it is the simply connected cover. Explicitly describing $Spin_n$ for $n \geq 8$ gets increasingly less insightful. There is a general definition that you can read about in §1.3 of Sepanski, *Compact Lie Groups*, but in general its smallest faithful representation is as a group of $N \times N$ matrices for N growing exponentially in n.

9. Consider the Lie group $\mathbb{R}_{>0} \ltimes \mathbb{R} \cong \left\{ \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \text{ s.t. } x \in \mathbb{R}_{>0}, y \in \mathbb{R} \right\} \subset GL_2(\mathbb{R})$. Show that the left-invariant measure is $x^{-2}\,\mathrm{d}x\,\mathrm{d}y$ whereas the right-invariant measure is $x^{-1}\,\mathrm{d}x\,\mathrm{d}y$. Conclude that the modular function for $\mathbb{R}_{>0} \ltimes \mathbb{R}$ is the projection onto the $\mathbb{R}_{>0}$ component.