MATH 4057/5057: Lie Theory

Assignment 2

suggested due date: February 9

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. Consider the element $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \in SU_2$, where $\lambda \in U_1$. Find the eigenvalues of the action of λ on $\mathrm{Sym}^n(\mathbb{C}^2)$. In particular, find a rational expression for the trace of the action of $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ in this representation.

2. Identify $\mathbb{C}^2$ with the space of linear functions in two variables x, y, so that $\mathrm{Sym}^n(\mathbb{C}^2)$ is then identified with the space of polynomials in x, y of homogeneous degree n. Consider the Hermitian form on $\mathrm{Sym}^n(\mathbb{C}^2)$ defined on a basis by:
$$\langle x^i y^{n-i} | x^j y^{n-j} \rangle = \delta_{ij} i!(n-i)!$$
Show that the standard action of SU_2 is unitary with respect to this Hermitian form.

3. Suppose that G is a Lie group and V, W are representations of G.
   (a) Show that the vector space of all linear maps $V \to W$ is naturally a G-representation. Let's write it as $\underline{\mathrm{hom}}(V, W)$.
   (b) In fact, show that $\underline{\mathrm{hom}}(V, W)$ carries a unique G-action such that the canonical map $\underline{\mathrm{hom}}(V, W) \otimes V \to W$ (defined on pure tensors by $t \otimes v \mapsto t(v)$) is a G-homomorphism. In other words, the category $\mathbf{Rep}(G)$ is *closed monoidal*.
   (c) Using only this canonical map $\underline{\mathrm{hom}}(V, W) \otimes V \to W$, show for any triple of objects V, W, X that $\mathrm{hom}(X, \underline{\mathit{hom}}(V, W)) = \mathrm{hom}(X \otimes V, W)$. In particular, writing $\mathbf{1}$ for the trivial G-representation, show that $\mathrm{hom}(\mathbf{1}, \underline{\mathrm{hom}}(V, W)) = \mathrm{hom}(V, W)$.

      In a monoidal category $\mathcal{C}$, a *global element* of an object $X \in \mathcal{C}$ is by definition a map $x : \mathbf{1} \to X$. This is shortened to "$x \in X$." So in the category $\mathbf{Rep}(G)$, the global elements in V are the G-fixed vectors (as opposed to all the vectors).
   (d) Define $V^* := \underline{\mathrm{hom}}(V, \mathbf{1})$. Show that there is a canonical G-homomorphism $V^* \otimes W \to \underline{\mathrm{hom}}(V, W)$.
   (e) Explain that if at least one of V, W is finite-dimensional, then this canonical comparison map $V^* \otimes W \to \underline{\mathrm{hom}}(V, W)$ is an isomorphism. Explain why it is not an isomorphism if both V, W are infinite-dimensional.

(f) Conclude that if V is finite-dimensional, then $V^* \otimes V \to \underline{\mathrm{hom}}(V, V)$ is an isomorphism. In a closed monoidal category, an object is called *dualizable* when $V^* \otimes V \to \underline{\mathrm{hom}}(V, V)$ is an isomorphism.

(g) Show that in a closed monoidal category, an object V is dualizable iff $\mathrm{id}_V \in \underline{\mathrm{hom}}(V, V)$ is in the image of the canonical comparison map $V^* \otimes V \to \underline{\mathrm{hom}}(V, V)$.

4. Consider the map $t \mapsto \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$.

   (a) Show that this map defines a representation of $\mathbb{R}$ on $\mathbb{C}^2$.

   (b) Show directly that this representation is not unitarizable.

   (c) Find all sub and quotient representations. In particular, conclude that this representation is indecomposable but not irreducible.

5. (a) Show that the defining representation of SO_2 on $\mathbb{R}^2$ is irreducible.

   (b) Show that its complexification, a representation of SO_2 on $\mathbb{C}^2$, is not irreducible, and decompose it as a sum of irreducibles.

6. Given n a positive integer, let us denote $\mathbf{n} := \mathrm{Sym}^{n-1}(\mathbb{C}^2) \in \mathbf{Rep}(SU_2)$. This is justified because $\dim(\mathbf{n}) = n$.

   (a) Show that $\mathbf{n} \cong \mathbf{n}^*$.

   (b) Show that $\mathbf{2} \otimes \mathbf{n} = [\mathbf{n} - \mathbf{1}] \oplus [\mathbf{n} + \mathbf{1}]$ if $n \geq 2$. What if $n = 1$?

   (c) Show that

$$\mathbf{m} \otimes \mathbf{n} = \bigoplus_{\substack{|m-n| \leq k \leq m+n, \\ k+m+n \in 2\mathbb{Z}}} \mathbf{k}.$$

   In other words, (k, m, n) should be the side-lengths of a valid triangle with even perimeter.