MATH 4057/5057: Lie Theory

Assignment 3

suggested due date: March 2

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. Let G be a finite group and M a finite set equipped with a permutation action of G. Consider the induced linear representation of G on $C(M)$. Show that the character of this representation counts the number of fixed points.

2. Let G be a group and V a finite-dimensional representation with character χ_V. Show that the characters of $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big),$$
$$\chi_{\mathrm{Alt}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 - \chi_V(g^2)\big).$$

3. A finite-dimensional unitary representation V of a compact group G is of *real type* if there is a real vector space $V_{\mathbb{R}}$ with an action of G and an isomorphism of representations $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V$. A finite-dimensional unitary representation V of a compact group G is of *quaternionic type* if there is a quaternionic module $V_{\mathbb{H}}$ and an isomorphism of G-representations between V and the underlying complex representation of $V_{\mathbb{H}}$.

   (a) Show that V is real type, resp. quaternionic type, if and only if it admits a G-invariant nondegenerate symmetric, resp. skew-symmetric, pairing.

   (b) Show that if V is irreducible, then the space of G-invariant nondegenerate pairings on V is either 0- or 1-dimensional. Conclude that V is exactly one of real type, quaternionic type, or *complex type* (when it is neither real nor quaternionic).

   (c) Show that if V is irreducible, then

$$\int_G \chi_V(g^2) = \begin{cases} +1, & V \text{ is real type,} \\ 0, & V \text{ is complex type,} \\ -1, & V \text{ is quaternionic type.} \end{cases}$$

   This number is called the *Frobenius–Schur indicator* of V.

4. Recall the notation: $\widehat{G}$ denotes the set of irreducible unitary representations of G; $\widehat{\bigoplus}$ denotes the ℓ^2-completed direct sum.

   (a) Let π_n denote the one-dimensional representation of $S^1 = \mathbb{R}/\mathbb{Z}$ given by $\pi_n(\theta) = \exp(2\pi i n\theta)$. Show that $\widehat{S^1} = \{\pi_n\}_{n\in\mathbb{Z}}$. Show that $\hom(\pi_n, L^2(S^1))$ is one-dimensional. Show that
   $$L^2(S^1) = \widehat{\bigoplus}_{n\in\mathbb{Z}}\pi_n.$$

   (b) Let ρ_y denote the one-dimensional representation of $\mathbb{R}$ given by $\rho_y(x) = \exp(2\pi i y x)$. Show that $\widehat{\mathbb{R}} = \{\rho_y\}_{y\in\mathbb{R}}$. Show that $\hom(\rho_y, L^2(\mathbb{R}))$ is zero-dimensional. Conclude that
   $$L^2(\mathbb{R}) \neq \widehat{\bigoplus}_{y\in\mathbb{R}}\rho_y.$$

5. Let G be a compact Lie group. Its *commutator subgroup* G' is the subgroup generated by elements of the form $xyx^{-1}y^{-1}$.

   (a) Show that $G' \subset G$ is normal and that G/G' is abelian.

   (b) Show that G' acts trivially on all one-dimensional representations of G.

   (c) Conclude that G is abelian iff all of its unitary irreps are one-dimensional.

6. Let G be a finite group. Explain why the formula
   $$|G| = \sum_{I\in\widehat{G}} \dim(I)^2$$
   decategorifies the Peter–Weyl theorem.

7. Let G be a finite group. Use the Peter–Weyl theorem to show that
   $$|\widehat{G}| = \dim(Z(C(G))) = |G/G|$$
   where $Z(C(G))$ is the centre of the convolution algebra of G and G/G denotes the quotient by the adjoint action.

8. Show that if G is compact and infinite, then $\widehat{G}$ is countable and infinite.

9. (a) Let V be an irrep of a compact group G. Show that
   $$\dim V \int_{g\in G} \chi_V(g^{-1}hgk) = \chi_V(h)\chi_V(k)$$
   for all $h, k \in G$.

   (b) Conversely, show that if $f : G \to \mathbb{C}$ satisfies $\dim V \int_{g\in G} f(g^{-1}hgk) = f(h)f(k)$, then $f = \chi_V$ for some irrep V.