# MATH 4057/5057: Lie Theory

Assignment 4

suggested due date: March 9

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. (a) Let $G$ be a Lie group. Show that $G$ admits a left-invariant metric.
   (b) Show that the exponential flow lines on $G$ are geodesics.
   (c) Show that in a compact connected Riemannian manifold, any two points are connected by a geodesic.
   (d) Conclude that if $G$ is a compact connected Lie group, then $\exp : \mathrm{Lie}(G) \to G$ is surjective.

2. Show that the compactness is necessary in the previous problem by showing that for $G = \mathrm{SL}_2(\mathbb{R})$, the exponential map is not surjective.

3. (a) A *bialgebra* is an associative algebra $(A, m, 1)$ equipped with an algebra homomorphism $\Delta : A \to A \otimes A$ which is coassociative and counital. The operation $\Delta$ is called the *comultiplication*. Show that if $A$ is a bialgebra, then the vector space $\mathrm{End}(A)$ of linear endomorphisms of $A$ is a monoid under the *convolution product* defined by $f \star g = m \circ (f \otimes g) \circ \Delta$. What is the unit with respect to convolution?

   Show that for a bialgebra, the following are equivalent:

   - The *shear map* $\mathrm{sh}_A : A \otimes A \to A \otimes A$ defined by $\mathrm{sh}_A = (\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A)$ is invertible with respect to composition.
   - $\mathrm{id}_A \in \mathrm{End}(A)$ is invertible with respect to convolution.

   If these equivalent conditions hold, then $A$ is called *Hopf*. The convolution-inverse to $\mathrm{id}_A$ is called the *antipode*.

   (b) Let $G$ be a discrete monoid. Show that the group algebra $\mathbb{K}[G] = \bigoplus_{g \in G} \mathbb{K}g$ is a bialgebra with $\Delta$ defined on generators by $\Delta g = g \otimes g$. In general, if $A$ is a bialgebra, an element $a \in A$ is called *grouplike* if $\Delta a = a \otimes a$. Show that $\mathbb{K}[G]$ is Hopf iff $G$ is a group, and describe the antipode.

   (c) Let $\mathfrak{g}$ be a Lie algebra. Show that its universal enveloping algebra $U\mathfrak{g}$ is a bialgebra with $\Delta$ defined on generators $x \in \mathfrak{g}$ by $\Delta x = x \otimes 1 + 1 \otimes x$. In general, if $A$ is a bialgebra, an element $a \in A$ is called *primitive* if $\Delta a = a \otimes 1 + 1 \otimes a$. Show that $U\mathfrak{g}$ is Hopf and describe its antipode.

(d) Show that the PBW isomorphism $\mathrm{Sym}^\bullet(\mathfrak{g}) \cong U\mathfrak{g}$ is an isomorphism of coalgebras. Conclude that the primitives in $U\mathfrak{g}$ are precisely the generators $\mathfrak{g} \subset U\mathfrak{g}$.

4. (a) Let $t$ be a formal variable and $A$ a bialgebra, and consider the formal power series ring $A[\![t]\!]$. Given $x \in A[\![t]\!]$, interpret "$\exp(tx)$" as an element in $A[\![t]\!]$. Show that $x$ is primitive iff $\exp(x)$ is grouplike. Conversely, suppose that $x \in A[\![t]\!]$ satisfies $x \equiv 1 \pmod t$; interpret "$\log(x)$" as an element in $A[\![t]\!]$ and explain why $\exp(-)$ and $\log(-)$ are inverse functions.

   (b) Let $s, t$ be two commuting formal variables, and consider the formal power series ring $A[\![s,t]\!]$. Suppose that $x, y \in A$ are primitive. Explain why

$$\log\bigl(\exp(sx)\exp(ty)\bigr)$$

   is primitive.

   (c) Consider now the case when $\mathfrak{g}$ is the free Lie algebra on two variables $x, y$ and $A = U\mathfrak{g}$. Explain that as an algebra, $A$ is the free associative algebra on two variables, i.e. it consists of all noncommutative polynomials in $x, y$. Explain that $\mathfrak{g}$ is the space of *Lie polynomials* in two variables i.e. iterated combinations of $[,]$. Conclude that there is a sequence of universal Lie polynomials so that

$$\log\bigl(\exp(sx)\exp(ty)\bigr) = sx + ty + \frac{st}{2}[x,y] + \frac{s^2t}{12}[x,[x,y]] + \frac{st^2}{12}[[x,y],y] + \dots$$

   and write out the next-degree terms.

   This is called the *Baker–Campbell–Hausdorff formula.*

   (d) * Prove that if $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{R}$, then for any $x, y \in \mathfrak{g}$, the Baker–Campbell–Hausdorff formula converges in some small ball in the $(s,t)$-plane. In other words, prove that without the $s, t$, this series converges in a neighbourhood of the identity in $\mathfrak{g}$.

5. (a) Let $F(d)$ denote the free Lie algebra on $d$ generators $x_1, \dots, x_d$. It has a natural $\mathbb{N}^d$ grading in which $F(d)_{k_1,\dots,k_d}$ is spanned by Lie brackets in which t$x_i$ appears $k_i$ many times. Use the PBW theorem to prove the following generating function identity:

$$\prod_{\vec{k}} \frac{1}{(1 - t_1^{k_1} \dots t_d^{k_d})^{\dim F(d)_{\vec{k}}}} = \frac{1}{1 - (t_1 + \dots + t_d)}.$$

   (b) A *primitive word* is a word in $x_1, \dots, x_d$ that is not a power of a shorter word. A *primitive necklace* is an equivalence class of primitive words under cyclic rotation. Use the generating function from part (a) to show that $\dim F(d)_{\vec{k}}$ is the number of primitive necklaces with $k_i$ occurrences of $x_i$.