MATH 4057/5057: Lie Theory

Assignment 5

suggested due date: March 30

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. Let G be a compact connected Lie group. Show that $Z(G)$ is the intersection of all maximal tori in G.

2. If S is a connected abelian subgroup of G, show that $Z_G(S)$ is the union of maximal tori containing S.

3. Let $T \subset SO_3$ be the maximal torus

$$T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \right\}.$$

   Find $g \in G$ such that T is equal to the connected component of $Z_G(g)$ but such that $Z_G(g)$ is disconnected.

4. A *octonions* $\mathbb{O}$ is the algebra $\mathbb{H} \oplus \ell\mathbb{H}$ with multiplication law $(x+\ell y)(x'+\ell y') = (xx' - y^* y') + \ell(x^* y' + yx')$, where $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ denotes the quaternion algebra and $(-)^*$ is quaternionic conjugation. The *octonionic conjugation* is $(x+\ell y)^* = x^* - \ell y$. The group G_2 is by definition the group of automorphisms of $\mathbb{O}$ as an algebra. Warning: $\mathbb{O}$ is nonassociative.

   (a) Show that if $z \in \mathbb{O}$ is *central* in the sense that $zw = wz$ for all $w \in \mathbb{O}$, then $z \in \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$.

   (b) Show that if $w \in \mathbb{O}$, then $w^* w$ is central.

   (c) Conclude that any $g \in G_2$ preserves the "inner product" $\langle u, v\rangle = \frac{1}{2}\big(u^* v + v^* u\big) = \frac{1}{2}\big((u+v)^*(u+v) - u^* u - v^* v\big)$. Explain that this inner product is positive definite and hence conclude that $G_2 \subset O_8$ is a closed subgroup of a compact group and hence compact.

   (d) Write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, and hence $\mathbb{O} = \mathbb{C} \oplus j\mathbb{C} \oplus \ell\mathbb{C} \oplus \ell j\mathbb{C}$. Show that $\mathbb{O}$ has a symmetry that is the identity on $\mathbb{O}$ and exchanges j with ℓ.

(e) Let r_ℓ denote the automorphism of $\mathbb{O} = \mathbb{H} \oplus \ell\mathbb{H}$ that is $+1$ on $\mathbb{H}$ and -1 on $\ell\mathbb{H}$. Let r_j denote the symmetry of $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ that is $+1$ on $\mathbb{C}$ and -1 on $j\mathbb{C}$, extended to $\mathbb{O}$ by declaring that $r_j(\ell) = \ell$. Let r_i denote the complex conjugation on $\mathbb{C}$, extended to $\mathbb{O}$ by declaring that $r_i(j) = j$ and $r_i(\ell) = \ell$. Observe that these three symmetries together generate a $A \subset (\mathbb{Z}/2\mathbb{Z})^3 \subset G_2$.

(f) Recall that $\mathrm{Aut}(\mathbb{H}) = SO_3$ is connected. Conclude that the subgroup of A generated by r_i, r_j is in the connected component of G_2. Use part 4d to conclude that the subgroup generated by r_i, r_ℓ is also in the connected component, and hence all of A is in the connected component.

(g) Show that $A \subset G_2$ is maximal abelian. Hint: any automorphism that commutes with r_ℓ will preserve the decomposition $\mathbb{O} = \mathbb{H} \oplus \ell\mathbb{H}$.

5. ! With notation as in the previous question, find a maximal torus of G_2. What is its rank? Hint: There is a map $Sp_1 = \{a \in \mathbb{H} \text{ s.t. } a^*a = 1\} \to G_2$ which sends $a \in \mathbb{H}$ to the automorphism $x + \ell y \mapsto axa^* + \ell ay$.

6. (a) Let G be a connected compact group. Show that $\mathrm{Lie}(G)$ decomposes canonically as $\mathfrak{z} \oplus \bigoplus_i \mathfrak{s}_i$ where $\mathfrak{z}$ is abelian and each $\mathfrak{s}_i$ is nonabelian simple. Let $Z, S_i \subset G$ denote the connected subgroup of G with Lie algebra $\mathfrak{z}, \mathfrak{s}_i$. Show that Z is the connected component of $Z(G)$, and hence closed, and that S_i is a closed subgroup.

(b) Show that the multiplication map $Z \times \prod S_i \mapsto G$ is a Lie group map with discrete, and hence finite central, kernel. Compute this map in the case $G = U_n$.

(c) Show that any closed normal Lie subgroup of G is (the image of) a product of some of the S_is with some central subgroup.

7. Let $N \subset GL_n$ denote the subgroup of $n \times n$ matrices, where over either $\mathbb{R}$ or $\mathbb{C}$, with 1s on the diagonal and zeros below the diagonal (and arbitrary values above the diagonal). Describe the Lie algebra $\mathfrak{n} \subset \mathfrak{gl}_n$. Show that the exponential map $\exp : \mathfrak{n} \to N$ is a global diffeomorphism. (What if the coefficients are in $\mathbb{H}$?)

8. Let $N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \subset GL_3$, and let $\Gamma \subset N$ denote the discrete subgroup generated by $\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$. Show that $\Gamma \subset N$ is normal. Show that the quotient group N/Γ has no faithful finite-dimensional representations.