MATH 4057/5057: Lie Theory

Assignment 6

suggested due date: April 20

Homework should be submitted as a single PDF attachment to `theojf@dal.ca`. Please title the file in a useful way, for example `Math4057_HW#_Name.pdf`.

You are encouraged to work with your classmates, but your writing should be your own. If you do work with other people, please acknowledge (by name) whom you worked with. You are expected to think about every problem on every assignment, but you are not expected to solve every problem on every assignment. This is an advanced class: you may need to look up terms, brush up on background, etc. The purpose of homework assignments is to learn.

1. Fix a positive root system $\Delta_+ \subset \Delta$. As in lectures, write $\alpha_1, \dots, \alpha_r$ for the simple roots and $\omega_1, \dots, \omega_r$ for their dual basis the fundamental weights. Use the killing form to identify the Cartan with its dual.

   (a) Show that $\rho = \sum_i \omega_i$ satisfies $r_{\alpha_i}\rho = \rho - \alpha_i$ for all simple roots α_i.

   (b) Show that $\rho' = \frac{1}{2}\sum_{\beta \in \Delta_+} \beta$ also satisfies $r_{\alpha_i}\rho' = \rho' - \alpha_i$ for all simple roots α_i.

   (c) Show that there the system of equations "$r_{\alpha_i}\rho = \rho - \alpha_i$" has a unique solution, and hence conclude that $\rho = \rho'$.

2. Correct the lecture: work out explicitly the root systems of the orthogonal Lie algebras $\mathfrak{so}_m$. As in the lecture, split this into even and odd cases, and polarize the basis as much as possible into chiral and antichiral lightlike directions. In other words, by changing to a complex basis, identify:

$$\mathfrak{so}_m \otimes \mathbb{C} \cong \{x \in \mathfrak{sl}_m(\mathbb{C}) : jx + x^T j = 0\}$$

   where

$$j = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

   In this basis, the diagonal matrices in this copy of $\mathfrak{so}_m \otimes \mathbb{C}$ are a Cartan, and I am asking you to work out how $\mathfrak{so}_m \otimes \mathbb{C}$ decomposes into eigenspaces over this Cartan. In this basis, the upper-triangular eigenvalues are a system of positive roots, and I am asking you to work out the corresponding Dynkin diagram.

   In particular, show that when $m = 2n$ you get a Dynkin diagram of type D_n, and when $m = 2n + 1$ you get type B_n.

   Deduce the isomorphisms $\mathfrak{so}_4 \cong \mathfrak{su}_2 \times \mathfrak{su}_2$, $\mathfrak{so}_5 \cong \mathfrak{sp}_2$, and $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. Deduce an exceptional automorphism of $\mathfrak{so}_8$.

3. A generalized Cartan matrix a is *affine* if $\det a = 0$ and all proper principal minors of a are positive. Classify the affine Cartan matrices.

   In particular, given an affine Dynkin diagram, if you remove any node, you get a compact Dynkin diagram. Show that, in the symmetric connected case, "remove a node" gives a bijection between the sets of affine and finite-type diagrams. Study the groups of Dynkin diagram automorphisms before and after removing a node.

   In the nonsymmetric case, the story is more complicated.

4. Let D be a finite-type symmetric (aka simply-laced) Dynkin diagram. Let $\Gamma \subset \mathrm{Aut}(D)$ be a subgroup of the automorphisms of D such that for every edge in D, the endpoints are in different Γ-orbits. The *folding* D/Γ is the diagram with a node for every Γ-orbit. Given Γ-orbits I, J, we put edge-weight k from I to J if each node of I is adjacent in D to k nodes of J.

   Show that the folding is again finite-type but not symmetric.

   Show that every finite-type Dynkin diagram is, in a unique way, a folding of a symmetric one.

5. Let L be an even lattice, i.e. Λ is a free abelian group equipped with a bilinear pairing $\langle , \rangle$ such that $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$.

   (a) Show that this already implies that $\langle \lambda, \lambda' \rangle \in \mathbb{Z}$ for all $\lambda, \lambda' \in \Lambda$.

   (b) Write e^Λ for the group Λ written multiplicatively. Show that there is a (nonabelian) extension

   $$\{\pm 1\} \to \hat{\Lambda} \to e^\Lambda$$

   such that the commutator $[e^{\hat{\lambda}}, e^{\hat{\lambda}'}] = (-1)^{\langle \lambda, \lambda' \rangle}$.

   (c) Show that any automorphism of Λ (preserving $\langle , \rangle$) lifts (noncanonically!) to $\hat{\Lambda}$.