T-duality for finite groups

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The context of my talk is the following. Suppose you have a group G, perhaps compact simple. And suppose you choose a class $\omega \in H^4(BG;\mathbb{Z})$. This isn't much extra data. See, most groups in the wild have a distinguished such class. For example, if G is compact simple, the $H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$ canonically, and you can take $\omega =$ positive generator.

In any case, let me also choose a finite order element $g \in G$, say order n. Let's assume $\omega|_{\langle g\rangle} = 0$. Consider the normalizer $N = N(\langle g\rangle)$ of $\langle g\rangle \cong C_n$. It, of course, fits into an exact sequence

$$C_n \cong \langle g\rangle \longrightarrow N := N(\langle g\rangle) \longrightarrow J := N/C_n.$$

In particular, as an extension $C_n . J$, N is determined by a 2-cocycle $\kappa \in H^2(BJ; C_n)$.

I want to ~~[illegible]~~ study the restriction $\omega|_N \in H^4(BN;\mathbb{Z})$. Since N is an extension, ~~[illegible]~~ $H^\bullet(N)$ can be analyzed by the Lyndon-Hochschild-Serre spectral sequence. The LHS s.s. is a type of "Taylor expansion": approximate $N \approx J \times C_n$; think of ω as a degree-4 function on N, and expand in the two coordinates.

It's traditional to draw spectral sequences in a grid:

LHS s.s.:

$$H^\bullet(J; H^\bullet(C_n)) \Rightarrow H^\bullet(C_n . J)$$

$$\begin{array}{r|lllll}
\cong C_n & H^p(J; H^4(C_n)) & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H^2(C_n) = \hat{C}_n & * & * & H^2(J;\hat{C}_n) & & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & \mathbb{Z} & * & * & * & H^4(J) \\
\hline
& & & & & \longrightarrow J
\end{array}$$

pontryagin dual

Recall what "there is a spectral sequence with this E_2 page and that ~~con~~ limit" means. It means that $H^4(N)$ has a filtration — it's an extension — whose pieces are subquotients of what you see in total degree 4 on the E_2 page. The fact that it's merely filtered is a typical thing from Taylor theory: it's the problem of extracting "Taylor coefficients" for a function on a smooth manifold — what you really have is the jet.

But in any case, for arbitrary $\varphi \in H^4(N)$, you certainly do have its "constant term" or "zeroth Taylor coeff" $\varphi^{(0)} = \varphi|_{C_n} \in H^4(C_n)$. In fact in $H^0(J; H^4(C_n))$. Ah, but our assumption was

$$\omega|_{C_n} = 0 .$$

When this happens, there is a well-defined class in $H^2(J; H^2(C_n)) = H^2(J; \widehat{C_n})$, — its next Taylor coeff. & In general, the leading, i.e. first non-zero, Taylor coeff is always well-defined. Let's call it

$$\alpha = \omega^{(2)} \in H^2(J; \widehat{C_n}) .$$

Now, there's one more datum in ω, namely the part in $H^4(J)$. Actually, it doesn't live in $H^4(J)$, but in some torsor for (some quotient of) it. Specifically, modulo certain "coboundaries", the last Taylor coeff $\beta = \omega^{(4)}$ lives in $C^4(J; \mathbb{Z})$ and solves not $d\beta = 0$ but

$$(*) \qquad d\beta = \square(\underbrace{\alpha \cup \kappa}_{\text{cup product, so lives in } H^4(J;\, \mathbb{Z}/n)})$$

κ ← original data of extension $N = C_n . J$.

$\square$: integral Bockstein $H^4(-;\mathbb{Z}_n) \to H^5(-;\mathbb{Z})$.

Now I can state the main point of my talk:

<u>Punchline:</u> Formula (*) is symmetric under $K \leftrightarrow \alpha$.

In particular, there is a "duality" operation on the set of data

$$\left\{ \left(N = C_n . J, \ \omega \in H^4(N) \text{ s.t. } \omega|_{C_n} = 0 \right) \right\}$$

$$\|$$

$$\left\{ \left(\breve{K}, \alpha, \beta \right) \text{ s.t. } \begin{array}{l} dK = d\alpha = 0 \\ d\beta = \square(\alpha \smile' K) \end{array} \right\}$$

$$\circlearrowright$$

$$K \leftrightarrow \alpha.$$

This is <u>finite group T-duality</u>. You ~~are~~ should think of C_n as a "finite circle" and the Pontryagin duality $C_n \leftrightarrow \widehat{C_n}$ as "inverting the radius of C_n". And ω is the "Kalb-Ramond field".

Years ago, when I was a graduate student, Kolya hosted a small dinner party for his students at the time. We had good food, plenty of wine, and then after dinner Kolya poured each of us a thimble full of vodka (which was plenty for me) and we asked him about conformal field theory. How did they know that Reshetikhin–Turaev theory (RT) was quantum Chern-Simons theory? By analyzing conformal field theories at the boundary.

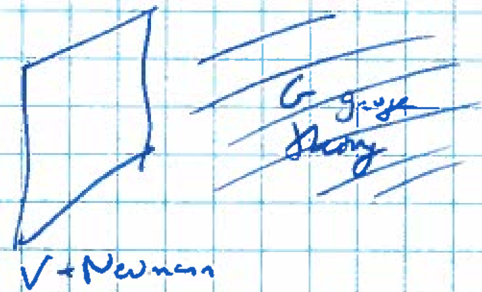
It took me another half decade, but I'm finally studying conformal field theories that live at the boundary of gauge theories.

In particular, suppose you have a 2D CFT, and an action of G on its algebra of observables. I'll call my CFT "V" for VOA, because the case I care most about are "holomorphic" CFTs, with trivial right-moving sector. In any case, an action $G \curvearrowright V$ might be anomalous — there might be an obstruction to gauging the action.

There is a general yoga that identifies

$$\{\text{CFTs w/ } G\text{-action}\} = \{\text{b.c.s for (pure) } G\text{-gauge thy}\}.$$

It goes as follows. If $V \circlearrowleft G$, then you can define V in the presence of a background ~~gau~~ G-bundle (gauge field). So you can form a bulk-boundary system w/ pure gauge theory in the bulk,



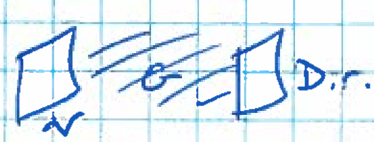
and declare that the gauge fields have Neumann boundary (arbitrary value), and couple to V via the symmetry.

The gauge theory involves a level in $H^4(BG)$ determining the Lagrangian for the thy (i.e. the "action" or measure of integration in the path integral). Anomaly cancelation says that the level in the bulk equals the 't Hooft anomaly at the boundary.

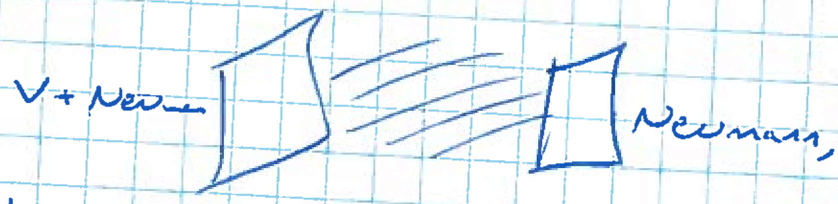
There is another way to say this. Look at the gauge-invariant boundary observables: you get the fixed subalgebra V^G. That algebra isn't holomorphic: it doesn't define a (full) CFT. Rather, it defines a b.c. for a RT theory build from the MTC $\mathrm{Rep}(V^G)$.

I care most about $|G| < \infty$. Gauge theories for $|G| < \infty$ are Dijkgraaf–Witten.

These thys always have a (pure) dirichlet boundary. If you stack, you reassemble V, because $N + D = \text{triv}$.



On the other hand, a pure Neumann boundary requires trivializing the anomaly-level. If you stack



the result is to gauge the 2D G-action, to get $V/\!/G$.

Caveat: this is a version of the construction of [Kirillov Jr] following [DVVV]. It requires that V^G be known to be rational. In the world of conformal nets this is known as soon as (V is holomorphic and) G is finite. But in the VOA world, the state of the art requires G to be solvable [CM].

In any case, a reinterpretation of ~~the~~ [END] says that:

__Theorem:__ Suppose V is holomorphic, $|G|<\infty$, and $G \circlearrowright V$ with anomaly $\omega \in H^4(BG)$. Let $g \in G$ s.t. $\omega|_{\langle g\rangle} = 0$.

Then the T-dual group to $N = N(g)$ acts on $V/\!/\langle g\rangle$, with the T-dual anomaly.

The most famous example of a finite group acting on a holomorphic VOA is the monster $\mathbb{M}$ acting on the moonshine module $V^\natural$. The monster is the largest sporadic group, with order

$$|\mathbb{M}| = \underbrace{2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 13^{3}}_{p\text{-Sylow nonabelian}}\cdot \underbrace{11^{2}\cdot 17\cdot \ldots \cdot 71}_{p\text{-Sylow is abelian}}$$

The Sylow structures are fairly systematic. For the abelian sylows $p = 11, 17, \ldots$, it is not hard to show

$$p\text{-part of } H^4(\mathbb{M}) = 0.$$

I want to calculate the anomaly ω of the action of $\mathbb{M}$ on $V^\natural$. It suffices to calculate its restriction to subgps containing the p-Sylows.

For the (nonabelian) $p = 2, 3, 5, 7, 13$, $\mathbb{M}$ has two distinguished (and various other) conj. classes of order p. Let $g \in$ second largest class, written

$$g = pB.$$

Then $N(g)$ contains the p-Sylow.

By [ALY],

$$V^\natural /\!\!/_{\langle pB \rangle} = V_{Leech}.$$

Now, the point is that T-duality mixes the data of $\omega|_{N(g)}$ with the data of the dual group $N(g)^\vee$.

But I know $N(g)$ and $N(g)^\vee$. So I know both "α" and "κ".

This knowledge, plus some computer computations, was what I used to prove

<u>Theorem</u>: The 't Hooft anomaly of the action of $\mathbb{M}$ on $V^\natural$ has order exactly 24.

<u>References</u>:

- This talk based on arXiv: 1707.08388.
- [ALY] arXiv: 1705.09022
- [CM] arXiv: 1603.05645
- [DVVV] MR1003430
- [ENO] arXiv: 0909.3140. MR2677836.
- [Kirillov] arXiv: math/0104242. MR1923177.
- [RT] MR1091619