Higher Galois closures

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Based on conversations with Mike Hopkins
and joint work in progress with David Reutter

Outline:

I. Fundamental thm of algebra and Tannakian duality

II. ∞-categorified commutative algebra

III. Galois group and j-homomorphism

These slides available at categorified.net/AGQFT.pdf

I

The Fundamental Theorem of Algebra (18th C) states that $\mathbb{C}$ is weakly terminal in the category of non-zero sufficiently-finite commutative $\mathbb{R}$-algebras.

e.g. finite dim separable.

$\exists A$, $\hom(A, \mathbb{C}) \neq \emptyset$.

More strongly, $\mathbb{C}$ is weakly terminal in the category of non-zero sufficiently-finite commutative $\mathbb{C}$-algebras.

(i.e. $\mathbb{C}$ is étale contractible.)

This selects $\mathbb{C}$ uniquely up to non-unique iso.

physicists like to label gps by their action.

Moreover, $\mathbb{R} \hookrightarrow \mathbb{C}$ is Galois with Galois gp $\mathbb{Z}_2^c = \{1, \text{c.c.}\}$:

$$\{\mathbb{R}\text{-linear mathematics}\} = \{\mathbb{C}\text{-linear mathematics} + \text{fixed point data for c.c.}\}.$$

Here is a weird fact about c.c.: $\lambda \mapsto \bar{\lambda}$ and $\lambda \mapsto \lambda^{-1}$ are homotopic as actions of $\mathbb{Z}_2$ on $\mathbb{C}^\times$.

$\longrightarrow$ in the analytic topology

Quantum mechanics escalates this weird fact by identifying c.c. with time reversal. "UNITARITY" "CRT thm"

TQFT manifestation: Consider the gpoid $\mathrm{Vec}_{\mathbb{C}}^{\sim}$ of finite dim $\mathbb{C}$-vector spaces. Two commuting actions of $\mathbb{Z}/2$:

- $V \mapsto \bar{V}$ c.c.   "algebraic"
- $V \mapsto V^*$ $O(1)$ "geometric"

Fixed-point data on V for the diagonal action $V \mapsto \bar{V}^*$ is a symmetric nondegenerate Hermitian form. The space of these has many components. One component, the positive definite forms, is contractible in the $\mathbb{R}$-topology.

Deligne's existence of super fibre functors (2002) states that $SVec_{\mathbb{C}}$ is weakly terminal in the 2-category

$\mathbb{Z}/_2$-graded v-spaces in which "commutative" ≡ "$yx = (-1)^{\deg x \deg y} xy$"

of non-zero sufficiently-finite symmetric monoidal $\mathbb{C}$-linear categories.

↪ separable multi-fusion. As in the $\mathbb{R} \subset \mathbb{C}$ case, stronger statements also hold.

More strongly, $SVec_{\mathbb{C}}$ is weakly terminal among non-zero sufficiently-finite sym mon $\mathbb{C}$-linear supercategories

SVec-enriched, aka "SVec-algebras"

This characterizes $SVec_{\mathbb{C}}$ uniquely but not canonically.

Moreover, $Vec_{\mathbb{C}} \hookrightarrow SVec_{\mathbb{C}}$ is Galois with

$$Gal(SVec_{\mathbb{C}}/Vec_{\mathbb{C}}) = \mathbb{Z}_2^f[1]$$

i.e. $B\mathbb{Z}_2$ with its gp str.
1-cell $\mapsto (-1)^f \in Aut_{\otimes}(id)$.

Let me spell this out a bit. If $\mathcal{C}$ is a sym $\otimes$ 1-cat$/\mathbb{C}$ then $\mathrm{Aut}_{\otimes,\mathbb{C}}(\mathcal{C})$ is a 2-group i.e. a gp object in 1-gpoids.

For $\mathcal{C} = \mathrm{SVec}_{\mathbb{C}}$, this 2-gp is connected, i.e. every auto is iso to id, but has $\pi_1 = \mathbb{Z}_2^f = \{1, (-1)^f\}$.

$(-1)^f$ is the natural transformation that assigns

$$\mathbb{C}^{1|0} \longmapsto +1 \qquad \mathbb{C}^{0|1} \longmapsto -1.$$

Category of fixed points is $\mathrm{Vec}_{\mathbb{C}}$.

$$\{\text{bosonic mathematics}\} = \{\text{super mathematics} + \text{fixed pt data for } (-1)^f\}.$$

i.e. $\mathrm{Vec}_{\mathbb{C}}$-enriched

Here is a weird fact about $(-1)^f$. $sVec_{\mathbb{C}}$ has two pivotal strs.

↪ monoidal natural iso $X \simeq X^{**}$.

For the pseudounitary one,

positivity condition →

$(-1)^f_X =$ 

Quantum field theory escalates this to an identification of $(-1)^f$ with $360°$ rotation. "Spin-statistics theorem"

TQFT manifestation: $\left\{ \begin{matrix} \text{fully-extended framed} \\ \text{2D super TQFTs} \end{matrix} \right\} = \left\{ \begin{matrix} \text{sep. superalgs} \\ \text{up to morita equiv} \end{matrix} \right\}$

has:

- $O(2) = \mathbb{Z}^b[1] \rtimes \mathbb{Z}_2^T$ action "geometric"
- $Gal(sVec_{\mathbb{C}}/vec_{\mathbb{R}}) = \mathbb{Z}_2^f[1] \rtimes \mathbb{Z}_2^c$ "algebraic"

The space of fixed-point data for the diagonal action has many components. One comp, the positive strs, is contractible.

in the $\mathbb{R}$-top.

II How to organize higher versions? A tower $\mathcal{C}^\bullet$ is [Scheimbauer] in the sense of Gaiotto–JF, see e.g. pirsa.org/20030111
a loop spectrum of higher categories. I.e.:

- a sequence $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2, \ldots$ s.t.

  $\mathcal{C}^n$ is an additive ($\oplus$'s) and Karoubi complete n-category equipped with a pointing, i.e. an object $1 \in \mathcal{C}^n$

- equivalences $\mathcal{C}^{n-1} \simeq \Omega \mathcal{C}^n := \mathrm{End}_{\mathcal{C}^n}(1)$.

As with loop spectra, each $\mathcal{C}^n$ is automatically sym. mon.

Example: $\Omega : \{\text{pointed } n\text{-cats}\} \to \{\text{monoidal } (n-1)\text{-cats}\}$ has a left adjoint Σ = Karoubi completion of one-object delooping.
$\rightsquigarrow$ suspension spectrum $\Sigma^\bullet \mathcal{V}$ of any sym $\otimes$ n-cat $\mathcal{V}$.

Example of example: $n\mathrm{Vec}_{\mathbb{K}} := \Sigma^n \mathbb{K}$.

$\Sigma^1 \mathbb{K} = \mathrm{Vec}$.
$\Sigma^2 \mathbb{K} = \mathrm{Alg}$.

A tower $\mathcal{C}^\bullet$ over $\mathbb{R}$ is sufficiently finite if each $\mathcal{C}^n$ is (fully) dualizable as an $\Sigma^n\mathbb{R}$-module.

Expectation (i.e. conjecture): This matches notions of separability from algebraic geometry.

Main conjecture: There exists an (ind-)sufficiently finite tower $\mathcal{R}^\bullet$ over $\mathbb{R}$ weakly terminal among nonzero sufficiently finite $\mathbb{R}$-linear towers.

i.e. ask that $\mathcal{R}^\bullet$ be "étale contractible"

If we add the stronger condition that $\mathcal{R}^\bullet$ is weakly terminal among nonzero sufficiently finite $\mathcal{R}^\bullet$-linear towers, then it is unique up to non-unique isomorphism. [Exercise]

If $\mathcal{R}^\circ$ exists, then $\mathcal{R}^n$ will be weakly terminal among symmetric monoidal n-categories.

<u>Pf</u>: Test against suspension towers.

$\mathbb{C}^\times$ $\quad \mathcal{R}^0 = \mathbb{C}$ [FTA]

$\mathbb{Z}_2$ $\quad \mathcal{R}^1 = \mathrm{SVec}_{\mathbb{C}}$ [Deligne]

$\mathbb{Z}_2$ $\quad \mathcal{R}^2 = \mathrm{SAlg}_{\mathbb{C}} = \Sigma\,\mathrm{SVec}_{\mathbb{C}}$ [Hopkins – JF, unpublished]

$\mathbb{Z}_{24}$ $\quad \mathcal{R}^3 =$ work in progress by Freed – Scheimbauer – Teleman on 3D TQFTs will probably build this.

in particular,

$\pi_{-n} I\mathbb{C}^\times = \hom(\pi_n \text{Spheres}, \mathbb{C}^\times)$

If $\mathcal{R}^\bullet$ exists, then

$\mathcal{R}^\times :=$ spectrum of invertible objects $\cong I\mathbb{C}^\times :=$ Pontryagin dual to spheres

<u>Pf</u>: Test against "group algebras".

So $\mathcal{R}^\bullet$ answers a request of Freed – Hopkins:

III I expect that the (ind-)separability of $\mathcal{R}^\circ$ will imply that $\Sigma^\circ\mathbb{R} \hookrightarrow \mathcal{R}^\circ$ is Galois.

Outlandish conjecture: $\mathrm{Gal}(\mathcal{R}^\circ/_{\Sigma^\circ\mathbb{R}}) \simeq O(\infty)$.

↑ issues of profinite completion and $\mathbb{R}$-topology.

Generalizing the CPT and Spin-statistics theorems, the algebraic $\mathrm{Gal} \simeq O(\infty)$ action should match the geometric $O(n)$ action on $\mathcal{R}^n$ coming from Cobordism Hypothesis.

Is this reasonable? Yes:

Pre-theorem [JF-Reutter]: On any symmetric fusion n-cat the $O(n)$-action enhances to an $O(\infty)$-action. The space of positive enhancements is contractible in the $\mathbb{R}$-topology.

The components of $\mathcal{R}^n$, denoted $\pi_0 \mathcal{R}^n$, are its simple direct summands as a $\Sigma\mathcal{R}^{n-1}$-module.

Outlandish conjecture suggests:

modulo issues of profinite completion and $\mathbb{R}$-topology...

Conjecture [Hopkins–JF–Reutter]:

$$\pi_0 \mathcal{R}^n \cong \text{Pontryagin dual to } \pi_n O(\infty).$$

$$\cup\!\mathsf{I}$$

components that contain an invertible object $=$ Pontryagin dual to image of $j : \pi_n O(\infty) \to \pi_n \text{Spheres}$.

And:

$$\underbrace{(\Sigma\mathcal{R}^{n-1})^\times}_{\text{invertibles in identity component}} = \text{Pontryagin dual to } \underbrace{\mathrm{coker}(j)}_{\text{the ``hard part'' of } \pi_n \text{Spheres}}.$$

Reasonable? Yes: Thm in progress (JF–Reutter): $\subseteq$.