Hypergroups and $\boxed{\text{Higher}}$ $\boxed{\text{fusion}}$ Categories

AIM 29 Sept 2022

I wasn't here, but apparently David mentioned hypergroups, and I'm supposed to do the sequel to his talk. Anyway, everything I say is jt w/ him.

Hypergps were first introduced ~~at the~~ ~1930, as a set w/ a multi-valued multiplication. The rule was that $a \cdot b$ is nonempty, and $\forall a\ \exists!\ a^{-1}$ s.t. $a \cdot a^{-1} \ni 1$.

A new version over $\mathbb{C}$, w/ $*$-str, came along by the late 20^{th} C. ~~to the present~~
The most modern version is in papers from the last decade by Van Daele and collaborators.

Let me start w/ the not quite fanciest version.
Pick a (com.) coef. ring R. A <u>discrete</u> hypergp over R consists of:

- a set G
- an R-alg str on RG, i.e. ~~s.t. $\forall a, b \in G$,~~ <u>fusion coefs</u> f_{ab}^c defined by $a \cdot b = \sum_c f_{ab}^c\, c$.
- s.t. $\forall a, b$, $\sum_c f_{ab}^c = 1$
  i.e. these coefs don't <u>count</u> the # of fusion channels, but rather measure the <u>probability</u> of a fusion channel.
- $\forall a$, $\exists!\ a^*$ s.t. $f_{aa^*}^1 \neq 0$.

rigid

Here's the main example. Take a (linear) monoidal n-category $\mathcal{C}$. This is a thing that allows you to draw <u>skein diagrams</u> in $n+1$ D, w/ objects of dimensions $0 \ldots n$. ~~[crossed out]~~ Look at the <u>skein space</u> of ~~S~~ $S^n_b = \partial D^{n+1} =$

$\subset \mathbb{R}^{n+1}$

What is this? It's a v-space spanned by the skeins that fit in this region (don't touch boundary) modulo local moves.

We can simplify this. Using rigidity, any skein is equal to a skein of shape

$x \in Ob(\mathcal{C})$

☆ some funny thing. Specifically, ☆$: 1 \to \int_{S^{n-1}_c} x$.

So $sk(S^n)$ is spanned by these.

Then there are relations, e.g. things like

<u>Note:</u> $Sk(S^n)$ is an assoc. alg!
Just stick a shell inside another shell.

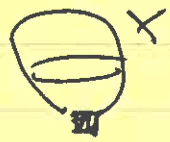
If $\mathcal{C}$ was more than monoidal, then $Sk(S^n)$ is a <u>comm</u> alg.

Anyway, now let me suppose $\mathcal{C}$ is fusion.
$\hookrightarrow$ Karoubian, locally semisimple, $< \infty$.

~~Thm:~~

Then, if X is simple, $\int_{S^{n-1}_{\mathcal{C}}} X = 1 \oplus \ldots$
so $\hom(1, \int_{S^{n-1}} X)$ is 1D, and we

Can choose a canonical rep by deciding to use the one s.t.

 $= 1$ if I fill in the interior.

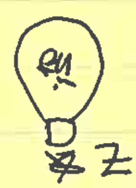
Moreover, if $X \overset{f}{\sim} X'$ two singles in same comp, then

pick s.t. $= 1$

by uniqueness of the thing.

<u>Conclusion:</u> $Sk(S^n_b)$ has a basis indexed by $\pi_0 \mathcal{C}$.

Moreover,

$1 =$  $= \sum_Z c^Z_{XY}$  $= \sum_Z c^Z_{XY}$

and this sum ranges are caps w/ a rep'n $X \otimes Y$

so this is a hypergp!

I think for the reasons I won't get to give the purely algebraic version. But I write it down in case I do.

~~A~~ ~~X~~ v-space w/ a basis $\mathcal{G}$

$\cap$|

coalg

by declaring the basis vectors to be gplike ($\Delta g = g \otimes g$, $\varepsilon g = 1$).

So hypgp $\subseteq$ { alg + coalg s.t. 1 is ~~coalg hom~~ gp like and ε is an alg hom }.

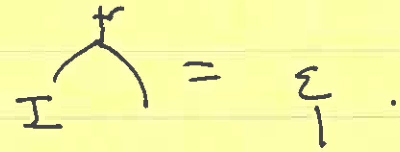
and two "gp" conditions, which can be expressed as follows. Pick the ~~(left?)~~ integral

~~$I \in A$ s.t.~~ map $A \to \mathbb{K}$

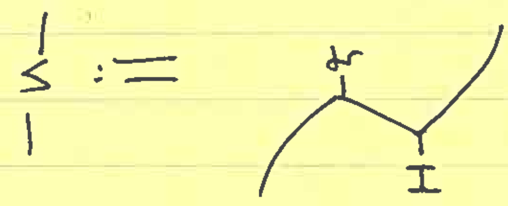
$1 \mapsto 1$

all rest $\mapsto 0$.

This is an "integral". I ask it to be a Frob. trace, and get the dual elt I defined by



then ask that the map.



is an iso, and an alg + coalg anti-map.

In the hopf case, this is $g \mapsto g^*$.

$$I = \sum_g \left(f_{g g^*}\right)^{-1} g \otimes g^* .$$

OK, so Dave asked me to explain the verlinde formula.

There are two versions.

First, let's just assume that A is comm + f.d. + semisimple (and $R = \mathbb{C}$.) Then A has a second basis, namely its spectrum $\hom(A, \mathbb{C})$ ✓ dual basis.

And this basis completely knows the alg. str:

$$A \underset{alg}{=} \bigoplus \mathbb{C} .$$

~~The splitting completely the~~

so the only data ~~of~~ of A give hypergps is the change of basis matrix.

$$\alpha(g) \quad \cancel{U}_{g,\cancel{\mu}} = \cancel{\mu}(g)$$

$$\left\{ U_{g,\alpha} \right\}_{\substack{g \in \text{gp tiles / vis basis} \\ \alpha \in \text{Spec}(A).}}$$

The multiplication?

$\alpha^\vee \in A$, dual basis to $\text{Spec}(A)$

$$a = \sum_\alpha U_{a,\alpha} \, \alpha^\vee$$

$$\alpha^\vee \cdot \beta^\vee = \delta_{\alpha,\beta} \, \alpha^\vee.$$

So
$$a \cdot b = \sum_\alpha U_{a,\alpha} U_{b,\alpha} \, \alpha^\vee.$$

$$\alpha^\vee = \sum_c (U^{-1})_{c,\alpha} \, c$$

so ~~$a \cdot b =$~~

$$f_{ab}^c = \sum_\alpha U_{a,\alpha} \, U_{b,\alpha} \, (U^{-1})_{c,\alpha}$$

~~N.B. Think ... = S~~

What is this $\mathcal{U}$?

Let's say two hypgps A, B are <u>paired</u> if I give you

$$S: A \otimes B \to \mathbb{K}$$

s.t. $A \to B^*$ is hom of hypgps

equiv $B \to A^*$ is hom of hypgps

equiv $A \to B^*$ and $B \to A^*$ are homs of algs.

best defn esp in inf. dims.

Spelled out:

~~Equiv~~, if $b \in B$ is gplike, then

$$S(aa', b) = S(a,b)\, S(a', b)$$

if $a \in B$ is gplike, then $S(a, bb') = S(a,b)\, S(a,b')$

Of course, if the pairing is nondeg, then $S = \mathcal{U}$. !

i.e. $A \xrightarrow{\sim} B^*$.

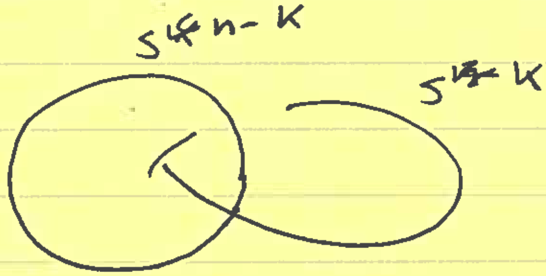
So how to get such S?

Suppose you have some fusion n-cat $\cancel{\mathcal{A}}\ \mathcal{C}^n$

Look at $\mathcal{A}^k = \Omega^{n-k} \mathcal{A}^n = $ (ops of dim $\leq k$)
This is a fusion k-cat.
Then our example of a hypergp was

$$\pi_{n-k} \cancel{\mathcal{C}} \cancel{\simeq} SK(S^k, \cancel{\mathcal{A}}\,\mathcal{C}^k) = A^k.$$

<u>But:</u>



$\subseteq \mathbb{R}^{n+1}$

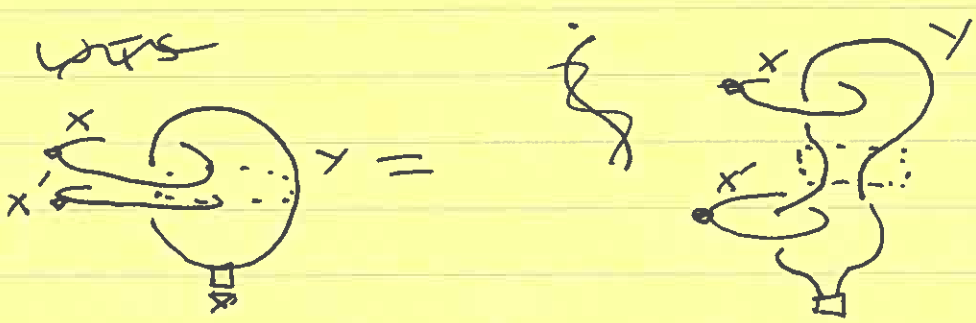
gives an obv. pairing $A^k \times A^{n-k} \to \mathbb{C}$.

might be deg.

<u>Claim:</u> this is a pairing of hypergps.

~~what do I need to show~~

<u>Pf:</u> ~~WTS~~

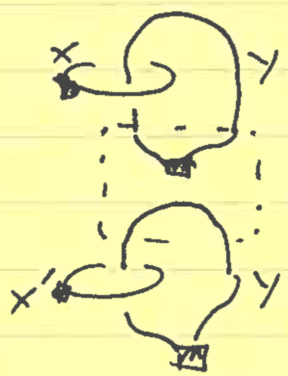


but this is one composition

$$\begin{array}{c} 1 \\ \uparrow \\ \int_{S^{k-1}} Y \\ \uparrow \\ 1 \end{array}$$

and $\int_{S^{k-1}} Y = 1 \oplus \dots$

so this factors ~~[illegible]~~ ~~as~~ thru 1:

$S =$ 

which is what we want to prove.

~~**Defn:** A fusion n-cat is nondegen iff all these S-matrices are~~

I should say: This pairing is what justifies the Δ on these hypergps.

Defn: Fusion n-cat $\mathcal{C}$ is nondeg if all these S-matrices are nondeg.

Thm (JFR): ~~[illegible]~~ nondeg $\Leftrightarrow$ Morita invertible.

i.e. if $\mathcal{C}$ = all ops in a TQFT (possibly w/ inv. anomaly).