0. Review of BV integration

Feynman: QFT = compute integrals of shape

$$\langle f \rangle = \int_{\mathcal{F}} f\, e^{-s/(i\hbar)}\, \mathrm{dVol}.$$

$\mathcal{F}$ = space of fields. $f \in \mathcal{O}(\mathcal{F})$ is "observable." $s \in \mathcal{O}(\mathcal{F})$ is "action." dVol usually DNE, as $\dim \mathcal{F} = \infty$.

de Rham: Integrals are controlled by homology: $\langle f \rangle$ depends only on class $[f\,\mathrm{dVol}] \in \mathrm{H}^{\dim \mathcal{F}}$ of *twisted de Rham complex* $\left(\Omega^\bullet(\mathcal{F}), \mathrm{d} - \frac{1}{i\hbar}\mathrm{d}s\wedge\right)$, by integration by parts.

As dVol DNE and $\dim \mathcal{F} = \infty$, use instead $\mathrm{MV}_\bullet(\mathcal{F}) = \mathcal{O}([1]\mathrm{T}^*\mathcal{F}) = \Gamma(\mathrm{T}^{\wedge\bullet}\mathcal{F})$ with differential $\Delta - \frac{1}{i\hbar}P(s,-)$. Δ = divergence w.r.t. dVol. P = Schouten–Nijenhuis bracket. Then $\langle f \rangle$ depends only on $[f] \in \mathrm{H}_0\big(\mathrm{MV}_\bullet(\mathcal{F}), \partial\big)$.

Application: Use formal nbhd of nondeg critical point of s, and $\hbar$ formal. Use differential $P(s,\cdot) - i\hbar\Delta$. Homology of $P(s^{(2)},\cdot)$ is one-dimensional. ($s^{(2)}$ = quadratic approximation of s.) Use homotopy perturbation theory to turn on "interaction term" $s - s^{(2)}$ and "integration term" $i\hbar\Delta$. You will end up reinventing Feynman diagrams.

1. The original AKSZ construction

Batalin–Vilkovisky: Δ is second-order differential operator w.r.t. $\wedge$ product in $\mathrm{MV}_\bullet(\mathcal{F})$. Its *principal symbol* $\Delta(a \wedge b) - \big(\Delta(a)\wedge b \pm a \wedge \Delta(b)\big)$ is P. Almost true:

$$\{\text{volume forms}\} = \{\Delta \text{ with princ. symb. } P \text{ s.t. } \Delta^2 = 0\}$$

N.B.: $P(s,-)$ = "contract with ds" is derivation of $\wedge$.

Defn: A *classical BV space* is a dg manifold $(\mathcal{E}, Q)$ with odd symplectic form ω of degree 1 in homological grading. (I.e. $\mathcal{E}$ = supermanifold with $\mathbb{G}_m$ action; Q = odd vector field of weight -1 for $\mathbb{G}_m$ action s.t. $Q^2 = \frac{1}{2}[Q,Q] = 0$; $\mathcal{L}_Q\omega = 0$.) **Example:** $\mathcal{E} = [1]\mathrm{T}^*\mathcal{F}$ = "extended space of fields," $\mathcal{F}$ any dg manifold.

A *formal deformation quantization* of $(\mathcal{E}, Q, \omega)$ is a second-order operator Δ of weight -1 defined over $\mathcal{O}(\mathcal{E})[\![\hbar]\!]$, such that $(Q - i\hbar\Delta)^2 = 0$ and Δ has principal symbol $P = \omega^{-1}$ = Poisson bivector. Note: if $\dim \mathcal{E} = \infty$, then ω^{-1} DNE without "renormalization theory."

Alexandrov–Kontsevich–Schwarz–Zaboronsky: To build a classical BV space, begin with:

- an "extended spacetime" M = dg manifold with volume form of weight d, and
- a "target" dg manifold X with symplectic form ω_X of weight $1-d$.

Set $\mathcal{E} = \underline{\mathrm{Maps}}(M,X)$ = infinite-dimensional dg manifold. $\mathrm{T}\mathcal{E} = \underline{\mathrm{Maps}}(M, \mathrm{T}X)$. Define:

$$\omega_{\mathcal{E}}(\vec{v}, \vec{w}) = \int_{m \in M} \omega_X\big(\vec{v}(m), \vec{w}(m)\big)$$

where $\vec{v}, \vec{w} \in \mathrm{T}_\varphi\mathcal{E}$, i.e. $\vec{v}(m), \vec{w}(m) \in \mathrm{T}_{\varphi(m)}X$.

Main examples: N = d-dimensional oriented manifold. $M = N_{\mathrm{htpy}} = \mathrm{spec}(\Omega^\bullet(N), \mathrm{d})$. $\underline{\mathrm{Maps}}(N_{\mathrm{htpy}}, X)$ = derived space of locally constant maps from N to X. Volume form $\int_M : \Omega^\bullet(N) \to \mathbb{R}$ is "integrate top forms."

Poisson Sigma Model (PSM): Y = Poisson manifold. $X = [-1]\mathrm{T}^*Y$ with dg structure computing Poisson homology. Canonical ω_X has weight -1. N = surface.

Chern–Simons Theory: $\mathfrak{g}$ = metric Lie algebra. $X = \mathrm{B}\mathfrak{g} = [-1]\mathfrak{g} = \mathrm{spec}(\mathrm{CE}^\bullet(\mathfrak{g}))$. Metric is equivalent to ω_X with weight -2. N = three-dimensional.

Also: BF Theory. Topological Yang–Mills. A-Model.

2. Factorization algebras and Poisson AKSZ

Costello–Gwilliam: QFT = factorization algebra (FA) of observables. To each region in spacetime assign dg vector space of "observables that can be measured within that region." Observables may be multiplied if they have disjoint support. Also a locality condition. **Example:** A "classical" field theory is a cosheaf of dg commutative rings.

Defn: A P_d *manifold* is a dg manifold with Poisson bracket of weight $d-1$. A *(semistrict) s.h. P_d manifold* is a graded manifold X with $S \in \mathcal{O}([-d]\mathrm{T}^*X)$ of weight $-d-1$ such that $\{S,S\} = 0$. *Flat* if $S|_X = 0$.

BV deformation quantization of FAs: Within space of FAs, can try to deform just dg structure. Any FA valued in (s.h.) P_0 algebras poses a deformation problem.

Poisson AKSZ Theory (manifold version): M still has volume form of weight d. X is s.h. P_d. Volume form gives dense inclusion $\underline{\mathrm{Maps}}(M, [-d]\mathrm{T}^*X) \hookrightarrow \mathrm{T}^*\,\underline{\mathrm{Maps}}(M,X)$. How to extend $(\varphi \mapsto \int_M \varphi^* S)$ to s.h. P_0 structure on $\underline{\mathrm{Maps}}(M,X)$ is problem of "renormalization theory."

Set $M = N_{\mathrm{htpy}}$ for N a region in d-dimensional spacetime. Then $N \mapsto \mathcal{O}(\underline{\mathrm{Maps}}(N_{\mathrm{htpy}}, X))$ is a *topological* FA. Homotopy pert. theory $\Rightarrow$ still TFA after deformation.

Francis, Lurie: Almost true: TFAs on d-dim spacetime = E_d algebras. (E_d = operad of little disks in $\mathbb{R}^d$.)

So quantization problem for Poisson AKSZ Theory = quantization problem P_d to E_d $\approx$ formality problem for E_d.

Relation to symp. version: Y s.h. P_d, then $X = [-d]\mathrm{T}^*Y$ is P_{d+1} with $Q = \{S,-\}$. X deforms to Hoch. coh. of def. of Y. Flat $\Leftrightarrow$ $Y \hookrightarrow X$ is Lagrangian; use Y as boundary brane for $(d+1)$-dim theory for X, e.g. PSM.

3. Lattice models of Poisson AKSZ Theory

More detail: AKSZ easiest when X = formal dg manifold in char 0. Then $X \cong$ vec. space, $\mathcal{O}(X) = \widehat{\mathrm{Sym}}(X^*_\bullet)$, and

$$\mathcal{O}(\underline{\mathrm{Maps}}(N_{\mathrm{htpy}}, X)) = \widehat{\mathrm{Sym}}\big((\Omega^\bullet(N)\otimes X)^*\big) = \widehat{\mathrm{Sym}}\big(\mathrm{Chains}_\bullet(N)\otimes X^*_\bullet\big)$$

for *some* dual space $\mathrm{Chains}_\bullet(N) = (\Omega^\bullet(N))^*$.

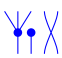
Usual choice in QFT: $\mathrm{Chains}_\bullet(N) = \Omega^{d-\bullet}_{\mathrm{cpt}}(N)$, perhaps distributional. **Other possibilities:** Use singular chains. Use triangulation of N. Choices correspond to models of N_{htpy} and of *topological chiral homology* = *factorization homology* of N valued in $\mathcal{O}(X)$. "Lattice AKSZ" = build theory from cell decomposition / triangulation.

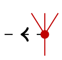
Slogan: Observables in AKSZ live on chains in N.

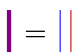
Remark: When X not formal, use more sophisticated lattice models. **E.g.:** usual Hochschild chains $\leftrightarrow N = $ .

Further user-defined input: $\widehat{\mathrm{Sym}}$ requires tensoring; must choose how with ∞-dim complexes. Canonical "choice": $\mathrm{Chains}_\bullet(N)^{\otimes 2} = \mathrm{Chains}_\bullet(N^2)$. But RHS has choices.

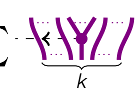
Warm-up: the dg structure: Functoriality of $\mathrm{Chains}_\bullet \Rightarrow$ pushforward along $N \hookrightarrow N^k$. Taylor-expand dg structure on X to $q^{(n)} : X^*_\bullet \to (X^*)^{\otimes n}_{\bullet-1}$. Useful notation:

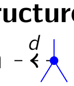
 = sample map $\mathrm{Chains}_\bullet(N^4) \to \mathrm{Chains}_\bullet(N^5)$.

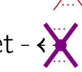
 $= q^{(3)}$ ( = shift in degree).

 $= \mathrm{Chains}_\bullet(N)\otimes X^*$. 

Combining gives dg structure on $\mathcal{O}(\underline{\mathrm{Maps}})$:

$$\partial|_{\mathrm{Sym}^k} = \partial|_{\mathrm{Chains}_\bullet(N^k)} \otimes \mathrm{id} + \mathrm{ave} \circ \sum$$ 

The Poisson structure: Recall N is oriented. So should exist intersection  : $\mathrm{Chains}_\bullet(N)^{\otimes 2} \to \mathrm{Chains}_{\bullet-d}(N)$, which with  gives Frobenius algebra. S.h. P_d str. on X has Taylor coeffs  : $(X^*)^{\otimes m} \to [d(m-1)-1](X^*)^{\otimes n}$.

Combine to get  and sum/average over choices.

Exercise: Strict Frob. alg. axioms + P_d axioms $\Rightarrow P_0$.

Slogan: For Poisson structure, bracket on X and $\cap$ in N.

4. Transversalization, renormalization, quantization

Renormalization problem: No model of $\mathrm{Chains}_\bullet$ with pushforward and intersections both strictly behaved, so we must modify the s.h. P_0 structure on $\underline{\mathrm{Maps}}$.

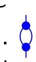
Defn: For most models of $\mathrm{Chains}_\bullet$, for each $S \subseteq \{1,\dots,k\}$ there is a quasi-iso subcomplex $\mathrm{Trans}^S_\bullet(N^k) \subseteq \mathrm{Chains}_\bullet(N^k)$ of chains *transverse to the "Sth diagonal"* $\{t_i = t_j,\ \forall i,j \in S\}$, where t_i is the coordinate on ith copy of N.

Basic fact: For any Frob. alg. axiom, let $S = \{$participating input indices$\}$. Then axiom holds on $\mathrm{Trans}^S_\bullet$.

Strategy: Homotope every chain to be transverse as required. A choice of system of *transversalizing homotopies* is our version of *renormalization scheme*.

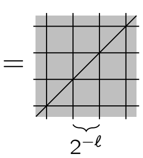
Theorem: Mild requests for transransversalizing homotopies $\Rightarrow$ sum of "tree-type" diagrams is s.h. P_0.

E.g.: In addition to , must also include  + sym., where ■ = transversalizing homotopy for $S = \{1,2,3\}$.

Theorem/Strategy: There is a similar sum of diagrams that may not be "tree-type." Provided further requests are satisfied (e.g.  $= 0$), this "quantum" sum gives a BV deformation of the classical theory.

5. Application: a universal rational $\star$-product

Theorem: Set X = universal (strict) P_1 formal manifold over $\mathbb{Q}$. Then the following model of $\mathrm{Chains}_\bullet$ supports a choice of transversalizing homotopies satisfying all requirements to give a BV quantization $\mathcal{F}^q$ of the AKSZ factorization algebra $\underline{\mathrm{Maps}}((\cdot)_{\mathrm{htpy}}, X)$ on $\mathbb{R}$:

$$\mathrm{Chains}_k(\mathbb{R}^n) = \bigcup_{\ell=0}^{\infty} \mathrm{Span}_{\mathbb{Q}}\Big\{ k\text{-cells in } \mathbb{R}^n = \Big\}$$ 

The $\star$-product: If $N \subseteq \mathbb{R}$ is contractible, can choose retraction $\int : \mathcal{O}(\underline{\mathrm{Maps}}(N_{\mathrm{htpy}}, X)) \leftrightarrows \mathcal{O}(X) : \iota$, where $\int$ = canonical map $\mathrm{Chains}_\bullet(N^k) \to \mathbb{Q}$ and ι = insert at some chosen point in N. HPT $\Rightarrow$ get deformed retraction $\tilde{\int} : \mathcal{F}^q(N) \leftrightarrows \mathcal{O}(X) : \iota$. Let $N_1 \cap N_2 = \emptyset$. FA axioms $\Rightarrow$ multiplication $\mathcal{F}^q(N_1)\otimes\mathcal{F}^q(N_2) \to \mathcal{F}^q(\mathbb{R})$. Choose N_1 to left of N_2, and set

$$\star = \tilde{\int} \circ \odot \circ (\iota_1 \otimes \iota_2) : \mathcal{O}(X)\otimes\mathcal{O}(X) \to \mathcal{O}(X).$$

Theorem: This $\star$-product is independent of choices of Ns and ιs and gives an associative multiplication satisfying $f \star g = fg + \frac{\hbar}{2}\{f,g\} + O(\hbar^2)$. It is defined over $\mathbb{Q}$.

Future work:

- Relation to PSM and to results by Kontsevich.
- With O. Gwilliam: Etingof–Kazhdan biquantization of Lie bialgebras; quantization of boundary branes.
- Non-formal (poly. in $e^\hbar$?) lattice AKSZ models, including Chern–Simons; study asymptotics as $\hbar \to 0$.