Separable and central simple (higher) algebras

ATCAT, 13 Oct 2020.

Defn: An adjunction in a 2-category (= bicategory, weak)

"$f \dashv g$"

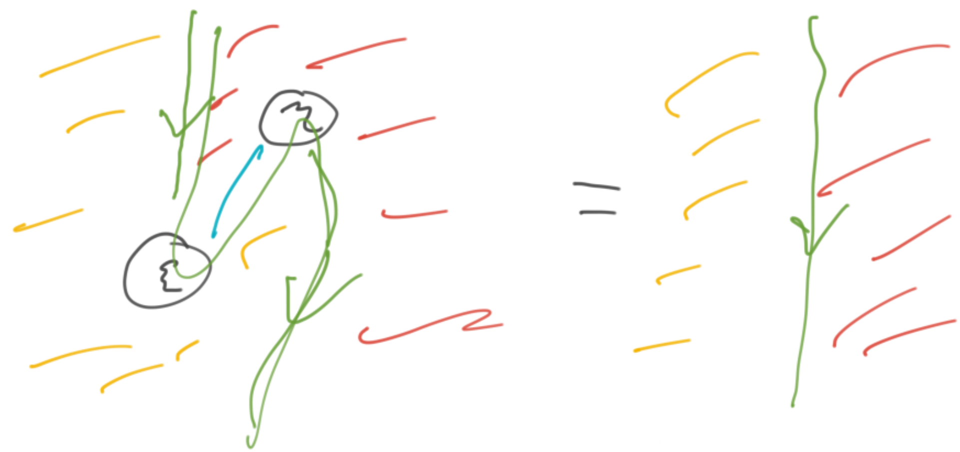
is:

$\boxed{Y} \xleftarrow{f} \boxed{X} \qquad \boxed{X} \xleftarrow{g} Y$

and 2-morphisms

$$\begin{array}{ccc} fg & & 1_X \\ \Downarrow \varepsilon & & \Downarrow \eta \\ 1_Y & & gf \end{array}$$

$$\begin{array}{c} f \\ \Downarrow f\eta \\ fgf \\ \Downarrow \varepsilon f \\ f \end{array} = 1_f \quad \text{and} \quad \begin{array}{c} g \\ \Downarrow \\ gfg \\ \Downarrow \\ g \end{array}$$



<u>Comment</u>: (*) In a 1-cat, $f \dashv g \iff f$ and g are inverses.

Adjunctibility is a possible categorification of invertibility. (**) $f \dashv g \not\Rightarrow g \dashv f$

<u>Exercise</u>: (1) Given f, there is a 1-category
objects are adjunction data (g, η, ε)
morphisms are ...
Show that if this cat is not empty then it is $\simeq \{*\}$.

(2) If f is invertible in the sense that $\exists g$ s.t. $fg \cong id$, $gf \cong id$, then I can choose these isos to be an adjunction

Look at 2-category $\mathrm{Mor}_{\Bbbk}$ ("morita")

Fix a com. ground ring $\Bbbk$

objects of $\mathrm{Mor}_{\Bbbk}$ are associative + unital $\Bbbk$-algebras

$$(B, A)\,\mathrm{mod} = \{B\text{-}A \text{ bimodules}\}$$

$$_C N_B \circ {}_B M_A = N \otimes_B M$$

which are the adjunctible 1-morphisms?

if $_B M_A \dashv {}_A N_B$ then $N \otimes_B - \cong \hom_B(M, -)$

preserves $\oplus$s, surjections.

So if ${}_B M_A$ is a left adjoint, $\Leftrightarrow$ M is f.g.

then $\hom_B(M, -)$ $\underbrace{\text{preserves } \oplus\text{s}}$, $\twoheadrightarrow$s.

So choose a presentation of M as a B-module

$$\begin{tikzcd}
\hom(M, B^{\oplus N}) \arrow[d, twoheadrightarrow] \\
\hom(M, M) \ni \mathrm{id}_M
\end{tikzcd}$$

So id_M is the image of a map $M \to B^{\oplus N}$

Conclusion: if M is right-adjunctible then it is a direct summand of a free B-module

"f.g. proj"

Converse easier.

Example: com. ring R e.g.

$$\mathrm{End}_{Mor}(R) = R\text{-modules w/ } \circ = \underset{R}{\otimes}$$

adjunctible ones = f.g. proj. modules =

invertible ones = line bundles.

e.g.: $\otimes$ 1-cat obj = LFP categories / R. $\mathrm{hom}(\mathcal{C}, \mathcal{D})$ = iso classes of colimit ... preserving functors.

Any $\otimes$ 1-category $\mathcal{C}$ "is" a 2-category

$B\mathcal{C}$ obj $= \{\bullet\}$, $\mathrm{End}_{B\mathcal{C}}(\bullet) = \mathcal{C}$

it can be interesting to ask about adjunctibility = "dualizability"

There remain interesting $\otimes$ categories where it is not known the classification of dualizable objects!

$\mathrm{Mor}_{\Bbbk}$ is a symmetric $\otimes$ 2-cat ← tensor product of algebras / $\Bbbk$.

What are the dualizable objects?] they all are.

if $A \in \mathrm{Mor}_{\Bbbk}$ is dualizable then

$$A^* \otimes - = \hom(A, -)$$

$$\boxed{h = 1}$$

$\rightsquigarrow$ left A^*-modules $\cong$ right A-modules

$A^* = A^{op}$

$$\begin{array}{c} A \otimes A^{op} \\ \varepsilon \downarrow \\ \Bbbk \end{array} \quad A \text{ as an right } A \otimes A^{op} \text{ module}$$

$$\begin{array}{c} \Bbbk \\ \eta \downarrow \\ A^{op} \otimes A \end{array} \quad A \text{ as a left } A^{op} \otimes A \text{-module}$$

In $\mathrm{Mor}_{\Bbbk}$ every object is dualizable

sym $\otimes$ 2-cat.

weak version of invertibility

$A \otimes A^* \xrightarrow{\varepsilon} 1 = \Bbbk$

$1 \xrightarrow{\eta} A^* \otimes A$

Because I'm in a 2-cat, intermediate between asking
- nothing of ε, η
- invertible

I could ask these 1-mors to be adjunctible.

Defn: Asking for this is called 2-dualizability of A.

invertible would have asked ε, η to be ISO

Which associative algebras are 2-dualizable?
$A \in \mathrm{Mor}_{k}$

$A^* = A^{op}$.

η, ε both "A as a bimodule between k and $A \otimes A^{op}$".

e.g. k = field
$A = L$
splits $\Leftrightarrow$ extension is separable.

We are asking

- ${}_{k}A_{A\otimes A^{op}}$ should be f.g. proj over k
"proper" "smooth"

"separable alg"

- A should be proj as an $A \otimes A^{op}$ module.
i.e. in cat of A-A bimodules

splits A-A linearly.

${}_A A \otimes A_A \xrightarrow{m} {}_A A_A$

Exercise: A choice of A-bilinear splitting

$$\begin{tikzcd} A \otimes A \arrow[d, "m"] \\ A \arrow[u, bend right, "\Delta"'] \end{tikzcd}$$

makes A into a Frob. alg.

in particular $\Delta : A \to A \otimes A$ is coassoc.

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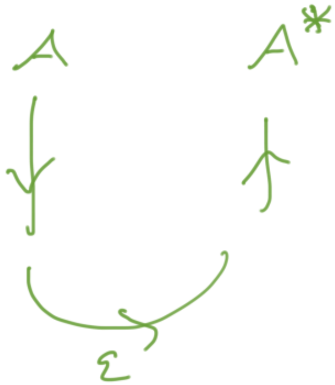
in $\mathrm{Mor}_{\Bbbk}$

- 1-dualizable = all
- 2-dualizable = f.d. sep. alg.
- invertible objects

  = central simple alg s.

  i.e. separable algs w/ $Z(A) = \Bbbk$.

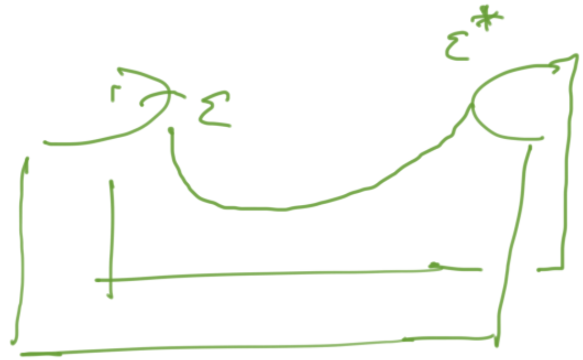


dualiability data

for ε, η

2-dualizability

let me draw

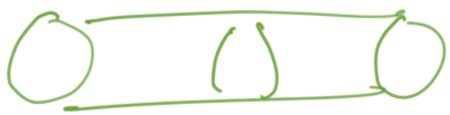


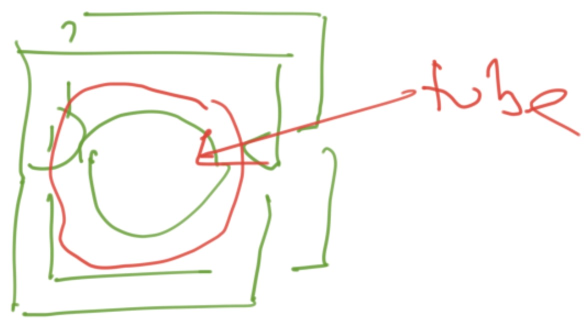
surfaces.

$$Z(A) = \hom\big(\underleftarrow{A_{A\otimes A^{op}}},\ \underbrace{A_{A\otimes A^{op}}}_{\varepsilon}\big) = \varepsilon^* \otimes \varepsilon.$$

Claim: if  $\Rightarrow$ empty

$\Leftarrow$

i.e. if  =  

then also 

is invertible

In $\boxed{\text{Mor}}$ 2-category: obj = monoidal vector spaces
"
0-cat.

- 1-dualizable = all objects
- 2-dualizable = sep. alg. } same. ← fusion categories higher.
- invertible = $\boxed{\text{central simple algs}}$ ← modular ⊗ categories

bundles w/ fibre $\mathbb{P}^n$.

e.g. over $\mathbb{R} = \mathbb{R}$, $\mathbb{H}$ is an example.

You could also ask about
a "morita n-category"
objects are monoidal $\boxed{\text{n-2 categories.}}$ ← what type of functor matters!

in one version of