Way too long. Think about how to cut down

Categorified Algebraic Closure

ATCAT 15 Feb 2022, Theo Johnson-Freyd

Given a com ring $\mathbb{K}$, recall that a com $\mathbb{K}$-algebra is simply a com ring R equiped with a map $\mathbb{K} \to R$.
Given a com $\mathbb{K}$-alg R and an element $r \in R$, we get a canonical map $\mathbb{K}[x] \xrightarrow{ev_r} R$, $x \mapsto r$.

<u>Defn:</u> $r \in R$ is <u>algebraic</u> over $\mathbb{K}$ if ev_r has kernel.
Algebra R is <u>algebraic</u> over $\mathbb{K}$ if every elt is.
In other words, r "solves a nontrivial poly equation".
In other words, algebraic algebras are built out of proper quotients $\mathbb{K}[x]/I$. Better: R is a colimit of rings of the form $\mathbb{K}[x]/(f)$ where you adjoin roots of polys.

A usual course on Galois thy is about such extensions where both $\mathbb{K}, R$ are fields. In a usual course, special emphasis is placed on the <u>separable</u> extensions, which is when $\forall r \in R$, "r solves an eqn w/ no repeated roots". Here is an equivalent condition: R is <u>separable</u> over $\mathbb{K}$ if the multiplication map

$$m : R \underset{\mathbb{K}}{\otimes} R \to R \qquad \text{i.e. } m\Delta = id_R$$

admits a one-sided inverse $\Delta : R \to R \underset{\mathbb{K}}{\otimes} R$ which is <u>R-bilinear</u>. — i.e. $\Delta(abc) = (a\otimes 1)\Delta(b)(1\otimes c)$.
In other words, R is <u>Frobenius</u>. (Exercise: Δ is coassoc.)
So we have categories

$$\text{sep'l exts} \subset \text{Algebraic extensions} \subset Com_{\mathbb{K}}$$

Initial object $= \mathbb{K}$. Terminal object $= 0$.

Remark: An alg geometer knows you should really work with $Com_{\mathbb{K}}^{op}$. Then $\mathbb{K} \mapsto$ "pt" is terminal in $Com_{\mathbb{K}}^{op}$.

This gives us a chance to ask: Given R, what is $\hom(R, \mathbb{K})$?

- algebrogeometrically, these are the "points" in R.
- algebraically, R is "a system of poly eqns" and these are the solns.

i.e. studying the functor $\hom(-, \mathbb{K}) : Com_{\mathbb{K}} \to Set$

In particular, usual situation is $R \neq 0$ but $\hom(R, \mathbb{K}) = \emptyset$.

But:

- ~~Sup~~ $\hom(R, \mathbb{K}) \neq \emptyset$ $\forall R \neq 0$ in {algebras}.
  iff $\mathbb{K}$ is an algebraically closed field.

Similarly, a field $\mathbb{K}$ is separably closed iff $\hom(R, \mathbb{K}) \neq \emptyset$ $\forall$ sep'l $R \neq 0$.

Let $\mathbb{K}$ be a field. Then $R \otimes_{\mathbb{K}} S \neq 0$ if $R, S \neq 0$, so $\otimes_{\mathbb{K}} S : \{\text{alg } com_{/\mathbb{K}}\} \to \{\text{alg } com_{/S}\}$ preserves nonzeros.
An algebraic closure (resp sep closure) of $\mathbb{K}$ is S alg closed. By adjunction,
$Aut_{\mathbb{K}}(S) \circlearrowright \hom_{\mathbb{K}}(R, S) = \hom_S(R \otimes S, S)$.

This lets you define a "Galois connection"
$$Com_{\mathbb{K}} \to Set_{Aut(S)}$$
which becomes an equiv for sep'l exts.

My goal is to tell you a categrification of this story. What do I mean?

$\boxed{\text{Commutative algebra} \rightsquigarrow \text{sym } \otimes \text{ category.}}$

I still want algebras to be abelian gps. So let's decide to work with <u>abelian</u> categories.

Small variations on "abelian" are OK, e.g. additive and finitely cocomplete w/o requiring im = coim.

An example should be $\mathrm{Mod}(\mathbb{K})$ for $\mathbb{K}$ a com ring, (Maybe f.g. mods.) with $\otimes = \otimes_{\mathbb{K}}$. So let's ask the "bilinearity of ~~$\otimes$~~ multiplication" to be that $\otimes : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is right-exact in each variable.

What do "algebraic extensions" look like? Let's focus on "ground ring" $\mathbb{K} \rightsquigarrow \mathrm{Vec}_{\mathbb{K}}$ where $\mathbb{K}$ is a ~~commutative~~ field of char zero.

~~Well,~~ So a "com $\mathrm{Vec}_{\mathbb{K}}$ alg" is a sym $\otimes$ abelian cat $\mathcal{R}$ ~~s~~ which is $\mathbb{K}$-linear.

What is the "polynomial ring" $\mathrm{Vec}_{\mathbb{K}}[x]$?

It is the free sym $\otimes$ ab. cat generated over $\mathrm{Vec}_{\mathbb{K}}$ by an object X. How does this look? It contains objects $1, X, X^{\otimes 2}, X^{\otimes 3}, \ldots$

but also: $X^{\otimes n}$ has an action by S_n.

i.e. $\mathrm{End}(X^{\otimes n}) = \mathbb{K}[S_n]$ gp alg.

I thank D. Reutter for explaining this to me.

In char $= 0$, we fully understand the rep thy of S_n: the simples are indexed by Young diagrams. Given a Young diagram λ, get an object of our abelian category X^λ. And $\mathrm{Vec}_{\mathbb{K}}[X]$ is semisimple w/ these simples.

Given $\mathbb{K}$-alg $\mathcal{R}$ and an object $V \in \mathcal{R}$, get functor $\mathbb{K}[X] \to \mathcal{R}$, $X \mapsto V$, and $X^\lambda \mapsto S^\lambda(V)$. Depends non-linearly monoidally on V. $S^\lambda(-)$ is called the λth <u>Schur functor</u>.

Let $\mathbb{K}$ a field of char $=0$.

<u>Defn</u>: Sym $\otimes$ ab $\mathbb{K}$-lin cat $\mathcal{R}$ is <u>algebraic</u> if $\forall V \in \mathcal{R}$, $\exists \lambda$ st. $S^\lambda(V) = 0$.

<u>E.g.</u>: Let G be a group (finite or inf, lie, alg, ...). Then $\mathrm{Rep}(G)$ is alg: if $V \in \mathrm{Rep}(G)$ has dim n, then $S^{\boxed{\vdots}_{\,n+1}} V = 0$. $S^{\boxed{\vdots}} = \Lambda^{n+1}$.

I actually don't know if Vec has an algebraic closure. To make progress, let's request a mild form of separability. Recall that $\mathcal{R}$ was sep if $\otimes$ had an $\mathcal{R}$-bilinear 1-sided inverse Δ. Let's replace "1-sided inverse" with "right adjoint".

Exercise: $\otimes : \mathcal{R}\boxtimes\mathcal{R}\to\mathcal{R}$ has an $\mathcal{R}$-bilinear right adjoint iff $\mathcal{R}$ is rigid ie. objects have duals. (This is slightly stronger than $\otimes$-closed. It is $\otimes$-closed and $\hom(X,Y) = \hom(X,1)\otimes Y$.)

Aside

Defn: Separable tensor cat is rigid and the counit $\otimes\circ\otimes^R : \mathcal{R}\to\mathcal{R}$, $\varepsilon\Downarrow$ $\mathrm{id}_{\mathcal{R}}$, has an $\mathcal{R}$-bilinear splitting $\mathrm{id}_{\mathcal{R}} \overset{\delta}{\Rightarrow} \otimes\circ\otimes^R$.

~~equiv the monad~~ ~~$\mathcal{R}\otimes\mathcal{R}$~~

equiv the monad $\otimes^R\circ\otimes$ on $\mathcal{R}\otimes\mathcal{R}$ is separable.

Thus I'll say that $\mathcal{R}$ is rigid-algebraic if it is rigid (mild separability) and $\forall V\in\mathcal{R}$, V is annihilated by a Schur functor.

skip

E.g./Proposition Suppose $\mathcal{R}$ is rigid. Then every $V\in\mathcal{R}$ has a dimension in $\mathbb{K}$. If V is annihilated by some Schur functor, then $\dim V\in\mathbb{Z}\subseteq\mathbb{K}$.

Thm (Deligne) Suppose $\mathbb{Q}\subseteq\mathbb{K}=\bar{\mathbb{K}}$ and $\mathcal{R}$ is rigid-algebraic and all dims are positive. Then $\hom(\mathcal{R},\mathrm{Vec})\neq\emptyset$.

Let's take $\mathbb{K} = \overline{\mathbb{K}} \supseteq \mathbb{Q}$.
Is $\mathrm{Vec}_{\mathbb{K}}$ alg closed?

No!

<u>Compare</u>: To go $\mathbb{R} \rightsquigarrow \mathbb{C}$, we start w/
$$\mathbb{R}[\mathbb{Z}/2] = \mathbb{R}[z]/(z^2=1),$$
and adjust to $\mathbb{R}[z]/(z^2=-1)$.

One category level up, we can do something similar as follows. Let's take
$$\mathrm{Vec}[\mathbb{Z}/2] = \mathrm{Vec}[Z]/Z^2 = \mathbb{1} = \oplus \mathbb{K}'$$

Now, this is sym $\otimes$, so it comes with the data of a map
$$\begin{array}{ccc} Z \otimes Z & = & \mathbb{1} \\ \times & & \\ Z \otimes Z & = & \mathbb{1} \end{array}$$

which implicitly we chose to be $+1$. But we could change our mind and set it to -1. The result is called sVec.

In other words, a general object of sVec is
$$V = V_0 \oplus V_1 Z$$
vector spaces say dims $= p, q$.

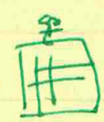
"$V = \mathbb{K}^{p|q}$"

i.e. as a cat, this is $\mathbb{Z}/2$-graded v-spaces.

For homogeneous elts.

The monoidal str is just the usual one:
$$\deg(v \otimes w) = \deg v + \deg w \mod 2.$$
The difference is in the braiding. The super braiding is
$$\mathrm{flip}(v \otimes w) = (-1)^{\deg v \deg w}\, w \otimes v.$$

E.g. $\mathrm{Dim}(\mathbb{K}^{p|q}) = p - q$.

$S^\lambda(\mathbb{K}^{p|q}) = 0$ if p  $< \lambda$.) if I recall correctly

Thm (Deligne) Let $\mathbb{K} = \overline{\mathbb{K}}$ alg closed field of char zero. Then $\mathrm{sVec}_{\mathbb{K}}$ is rigid-algebraically closed: if $\mathcal{R} \neq 0$ is a (finitely generated) $\mathrm{sVec}_{\mathbb{K}}$-alg which is rigid-alg, then $\exists\ \mathcal{R} \to \mathrm{sVec}_{\mathbb{K}}$.

(In fact, there is an algebraic stack "$\mathrm{Spec}(\mathcal{R})$" s.t. $\mathcal{R} = \mathrm{sVec}$-valued qcoh $(\mathrm{Spec}\,\mathcal{R})$.

Let's try Galois thy.

$$\mathrm{Gal}(\mathrm{sVec}/\mathrm{Vec}) = \mathrm{Aut}_{\mathrm{sym}\otimes}(\mathrm{sVec}) = ??$$

Well an elt is a sym $\otimes$ functor $F: sVec \to sVec$. ($\mathbb{K}$-lin)
It preserves $\mathbb{1}$ and sends $Z \mapsto$ something iso to Z.
So as a functor, it is iso to the identity.
To be a sym $\otimes$ functor, you need to supply the data of

$$F(X \otimes Y) \simeq F(X) \otimes F(Y)$$

natural in X, Y and compat w/ associativity + commutativity isos. Suffice to supply

$$F(Z \otimes Z) \simeq F(Z) \otimes F(Z)$$

$F(Z \otimes Z) = F(\mathbb{1}) = \mathbb{1}$, and $F(Z) \otimes F(Z) \cong Z \otimes Z = \mathbb{1}$. (choose trivialization $F(Z) \simeq Z$)

i.e. a number α in $\mathbb{K}^\times$. But by changing the triv $F(Z) \simeq Z$ by β, can adjust $\alpha \mapsto \alpha\beta^2$. So since $\mathbb{K} = \overline{\mathbb{K}}$, there is only one such F up to iso.

But that's not the end of the story!
This is a cat: F can have automorphisms.
And indeed: if you adjust by $\beta = -1$, you get an auto of F.

$$\boxed{\text{Prop: } Gal(sVec / vec) = B\mathbb{Z}_2 \quad \text{i.e. the gp object in Cat.}}$$

Moreover, $\mathrm{Vec} \hookrightarrow \mathrm{sVec}$ is Galois.

Thm (Deligne's super Tannakian duality, finite case)

Suppose $\mathcal{R}$ is a finite ~~~~ separable comm Vec-alg.

Then there is a finite gpoid

$$\hom(\mathcal{R}, \mathrm{sVec}) = \mathcal{X}$$

with an action by $\mathrm{Gal}(\mathrm{sVec}/\mathrm{Vec}) = B\mathbb{Z}/2 = \mathcal{D}$

and the canonical map

$$\mathcal{R} \longrightarrow \{\mathcal{D}\text{-equivariant functors } \mathcal{X} \to \mathrm{sVec}\}$$

is an equivalence of sym $\otimes$ cats.

E.g.: $\pi_0 \mathcal{X} = \mathrm{Spec}(\mathrm{End}_{\mathcal{R}}(\mathbb{1}))$. So $\mathcal{X} = BG$ if $\mathbb{1} \in \mathcal{R}$ is simple.

In the f.g. rigid-algebra case, ~~~~

$\mathcal{X} = BG$ with G an affine alg super gp scheme (of finite type).

done.