Algebraic definition of topological order

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These slides: categorified.net/AlgTopOrder.pdf.

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Kong–Wen et al: $(n+1)$D topological order = fusion n-category with remote detectability.

Goal for my talk: Complete mathematical definition.

Plan for my talk:

Weak n-categories

Categorical condensations

(Separable) multifusion n-categories

Remote detectability

Fusion and braided fusion n-categories

Classification of topological orders

Weak n-categories

Defn (part i of iii): We weak 0-category is a set. A weak n-category is an $(\infty, 1)$-category enriched in the $(\infty, 1)$-category of weak $(n-1)$-categories.

This means that $\mathcal{C}$ consists of the following data:

- A set $\mathrm{ob}(\mathcal{C})$ of objects (aka 0-morphisms) of $\mathcal{C}$.
- For each $(k+1)$-tuple $(X_0, \dots, X_k)$ of objects of $\mathcal{C}$, a weak $(n-1)$-category $\mathcal{C}(X_0, \dots, X_k)$. This is the collection of composable k-tuples $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{\cdots} X_{k-1} \xrightarrow{f_k} X_k$.
- Simplicial structure: strict functors

$$\text{composition} = \text{delete } X_i : \mathcal{C}(\dots, X_i, \dots) \to \mathcal{C}(\dots, \widehat{X}_i, \dots)$$
$$\text{insert identity} = \text{repeat } X_i : \mathcal{C}(\dots, X_i, \dots) \to \mathcal{C}(\dots, X_i, X_i, \dots)$$

These data must satisfy the Segal axiom on next slide.

Defn (part ii of iii): These data must satisfy the Segal axiom:

$$\mathcal{C}(X_0, \ldots, X_k) \to \prod_{i=1}^{k} \mathcal{C}(X_{i-1}, X_i)$$

is a weak equivalence (see part iii) of $(n-1)$-categories.

Upshot:

$$\mathcal{C}(X_0, X_1) \times \mathcal{C}(X_1, X_2) \overset{\sim}{\leftarrow} \mathcal{C}(X_0, X_1, X_2) \to \mathcal{C}(X_0, X_2)$$

gives a contractible space of composition maps. You could, if you want, choose some noncanonical splitting to get a composition $\circ : \mathcal{C}(X_0, X_1) \times \mathcal{C}(X_1, X_2) \to \mathcal{C}(X_0, X_2)$.

Associativity from compatibility of $\mathcal{C}(X_0, X_1, X_2, X_3) \to \mathcal{C}(X_0, X_3)$. Higher associativity data from $\mathcal{C}(X_0, \ldots, X_k) \to \mathcal{C}(X_0, X_k)$.

Defn (part iii of iii): It is clear what is a strict functor $F : \mathcal{C} \to \mathcal{D}$ of weak n-categories. A strict functor is a weak equivalence if it is

- Fully faithful: All $\mathcal{C}(X_0, \ldots, X_k) \to \mathcal{D}(FX_0, \ldots, FX_k)$ are weak equivalences.
- Essentially surjective: Every object of $\mathcal{D}$ is isomorphic to an object in the image of F.

If you like such things, you can make (weak n-categories, weak equivalences) into a model category. The details don't matter for most users: I mention it just so that you sleep easy.

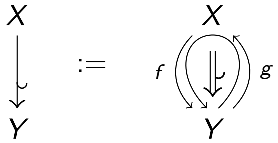
At the end of the day, weak n-categories have i-morphisms for $i \leq n$. The n-morphisms form sets, and their composition is strict. i-morphisms for $i < n$ do not have strict composition.

Can just as easily define $\mathbb{C}$-linear weak n-category, in which the n-morphisms form $\mathbb{C}$-vector spaces, and compositions are bilinear.

Categorical condensations

All that said, a lot of the time you can just ignore all associator and homotopy stuff, especially whenever you are studying structures parameterized by n-computads (free weak n-categories) which are gaunt (all isomorphisms are equalities).

Example: A condensation $X \rightarrow Y$ [1] in a weak n-category $\mathcal{C}$ is a pair of 1-morphisms $f : X \leftrightarrows Y : g$ and a condensation $fg \rightarrow \mathrm{id}_Y$. These are parameterized by a gaunt computad



Condensations are n-cat version of split surjection aka retract.

[1] LaTeX: `\mathrel{\,\hspace{.75ex}\joinrel\rhook\joinrel\hspace{-.75ex}\joinrel\rightarrow}`

A condensation monad aka (nonunital) separable monad is an endomorphism $e : X \to X$ plus an associative condensation $e^2 \twoheadrightarrow e$. Condensation monad e splits if it factors $e = gf$ through a condensation $f : X \twoheadrightarrow Y : g$. $\mathcal{C}$ is condensation complete aka Karoubi complete if all condensation monads split.

Theorem (Gaiotto–JF, Douglas–Reutter for $n = 2$):

(1) If a condensation monad splits, then the splitting is unique.

(2) There is a natural construction $\mathcal{C} \rightsquigarrow \mathrm{Kar}(\mathcal{C})$ that condensation-completes any $\mathcal{C}$.

(3) Condensation complete $\Rightarrow$ complete for all absolute colimits.

(4) $n\mathrm{V{\small EC}} := \Sigma^{n-1}\mathrm{V{\small EC}} \subset \{$cond complete linear $(n-1)$-cats$\}$ is the fully-dualizable subcategory. **Notation:** If $\mathcal{C}$ is monoidal, $\Sigma\mathcal{C} := \mathrm{Kar}($one-point delooping of $\mathcal{C})$. $\mathrm{V{\small EC}}$ = f.d. vspaces.

Caveat: Full story of colimits in enriched higher cats still under development. (3,4) assume it will be "the same" as classical story.

(Separable) multifusion *n*-categories

Recall: A multifusion 1-category $\mathcal{A}$ is a monoidal $\mathbb{C}$-linear Karoubi-complete category which is

(1) semisimple with finitely many simples;

(2) rigid: all objects in $\mathcal{C}$ have duals.

Definition of multifusion 2-category due to **Douglas–Reutter**.

How to find correct *n*-categorical generalization?

Tillmann: (1) $\Leftrightarrow$ $\mathcal{A}$ is 1-dualizable $\Leftrightarrow$ $\mathcal{A}$ is fully dualizable in $\mathrm{KarCat}_{\mathbb{C}} := \{$Karoubi complete $\mathbb{C}$-linear cats$\}$, i.e. $\mathcal{A}$ is proper.

Exercise (Gaitsgory): (2) $\Leftrightarrow$ tensor product $\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ has an $\mathcal{A}$-bilinear right adjoint $\Delta = \otimes^R : \mathcal{A} \to \mathcal{A} \boxtimes \mathcal{A}$. In particular, counit of adjunction $\eta : \otimes \circ \Delta \Rightarrow \mathrm{id}_{\mathcal{A}}$ is $\mathcal{A}$-bilinear.

Douglas–Schommer-Pries–Snyder: Since $\mathrm{char}(\mathbb{C}) = 0$, there exists an $\mathcal{A}$-bilinear splitting ϵ s.t. $\eta\epsilon = \mathrm{id}_{\mathrm{id}_{\mathcal{A}}}$, i.e. $\mathcal{A}$ is separable aka smooth. I.e. $\otimes : \mathcal{A} \boxtimes \mathcal{A} \twoheadrightarrow \mathcal{A}$ is an $\mathcal{A}$-bilinear condensation.

Defn: A (separable) multifusion n-cat $\mathcal{A}$ is a monoidal (aka E_1) $\mathbb{C}$-linear Karoubi-complete n-category which is

proper: $\mathcal{A}$ is 1-dualizable (in fact, fully dualizable) in $n\mathrm{KarCat}_{\mathbb{C}} := \{$Karoubi complete $\mathbb{C}$-linear n-cats$\}$.

smooth: multiplication map $\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ extends to an $\mathcal{A}$-bilinear condensation.

Theorem: For a monoidal $\mathbb{C}$-linear Karoubi-complete n-category $\mathcal{A}$, the following are equivalent:

- $\mathcal{A}$ is (separable) multifusion.
- $\mathcal{A}$ is 2-dualizable in the $(n+2)$-category $\mathrm{Mor}_1(n\mathrm{KarCat}_{\mathbb{C}})$.
- $\mathcal{A}$ is fully dualizable in $\mathrm{Mor}_1(n\mathrm{KarCat}_{\mathbb{C}})$.

The corresponding $(n+2)$D TFT is what X.G. Wen calls the anomaly of the $(n+1)$D topological order with excitations $\mathcal{A}$.

JF–Scheimbauer: Construction of $\mathrm{Mor}_1(n\mathrm{KarCat}_{\mathbb{C}})$.

Remote detectability

Defn: The (Drinfeld) centre $Z(\mathcal{A})$ of $\mathcal{A}$ is the *n*-category of $\mathcal{A}$-bimodule endomorphisms of $\mathcal{A}$. It is automatically $\mathbb{C}$-linear Karoubi complete.

$\mathcal{A}$ satisfies remote detectability if $Z(\mathcal{A}) \simeq n\mathrm{V\small{EC}}$, i.e. trivial.

Theorem: For a multifusion *n*-category $\mathcal{A}$, the following are equivalent:

- $\mathcal{A}$ satisfies remote detectability.
- $\mathcal{A}$ is invertible in the $(n+2)$-category $\mathrm{M\small{OR}}_1(n\mathrm{K\small{AR}}\mathrm{C\small{AT}}_{\mathbb{C}})$.

Defn: An (unstable) $(n+1)$D algebraic topological order is a multifusion *n*-category satisfying remote detectability.

Corollary:

$$\{(n+1)\text{D algebraic topological orders}\} = \frac{\{(n+1)\text{D TFTs}\}}{\{\text{invertible TFTs}\}}$$
$$= \{(\text{gravitationally}) \text{ anomalous framed } (n+1)\text{D TFTs}\}$$

Fusion and braided fusion *n*-categories

Notation: If $\mathcal{A}$ is a multifusion n-category, write $\Omega\mathcal{A}$ for the endomorphism $(n-1)$-category the unit object $1_{\mathcal{A}} \in \mathcal{A}$. It is automatically braided (aka E_2) multifusion.

Physics: $\mathcal{A}$ is the n-category of codimension-(≥ 1) excitations in some topological order. $\Omega\mathcal{A}$ is the codimension-(≥ 2) excitations. Continue: $\Omega^k\mathcal{A}$ is the codimension-$(\geq k)$ excitations.

Recall: A multifusion 1-category $\mathcal{A}$ is fusion if $\Omega\mathcal{C} = \mathbb{C}$. Equivalently, $1_{\mathcal{A}}$ does not decompose as a nontrivial direct sum.

Defn: A multifusion n-category is fusion if $\Omega^n\mathcal{A} = \mathbb{C}$. Equivalently, $1_{\mathcal{A}}$ does not decompose as a nontrivial direct sum.

Physics: $\Omega^n\mathcal{A}$ is a commutative separable finite-dimensional $\mathbb{C}$-algebra, so $= \mathbb{C}^{\bigoplus N}$ for some $N < \infty$. In high-energy QFT, $\mathrm{Spec}(\Omega^n\mathcal{A})$ are the N local vacua. If $N > 1$, then the system is unstable: for each local vacuum, there is a small operator that you can add to the Hamiltonian which projects onto that vacuum.

Theorem: For $\mathcal{A}$ an (n+1)D (possibly unstable) algebraic topological order, i.e. a multifusion n-category satisfying remote detectibility, the following are equivalent:

(1) $\mathcal{A}$ is fusion, i.e. $\Omega^n \mathcal{A} = \mathbb{C}$.

(2) $\mathcal{A} = \Sigma\Omega\mathcal{A}$.

Remark: (2) Says that there are no "nontrivial" codimension-1 operators. But interpret this carefully! There are typically lots of codimension-1 operators, including many that do not decompose as a direct sum. What makes them "trivial" is that they all arise from condensing codimension-(≥ 2) operators.

Main step in proof: More generally, suppose $\mathcal{A}$ is fusion but not necessarily remote detectable. Then $\rho : Z(\mathcal{A}) \to \mathcal{A}$ is dominant: every object $Y \in \mathcal{A}$ is the image of a condensation $X \twoheadrightarrow Y$ with $X \in \mathrm{image}(\rho)$. This gives (1 $\Rightarrow$ 2), and (2 $\Rightarrow$ 1) is easy.

More generally:

Theorem: For $\mathcal{A}$ an $(n+1)$D algebraic topological order, the following are equivalent:

(1) $\Omega^{n-k}\mathcal{A} = k\mathrm{Vec}$.

(2) $\mathcal{A} = \Sigma^{k+1}\Omega^{k+1}\mathcal{A}$.

Slogan: If all excitations of dimension $\leq k$ are "trivial," then all morphisms of codimension $\leq k+1$ are "trivial."

Outline of proof: The hard direction is (1 $\Rightarrow$ 2). Define E_{k+1}-centre $Z_{(k+1)}$ (e.g. E_2-centre = Müger centre). Without assuming remote detectability, show that if $\mathcal{B}$ is an E_{k+1}-monoidal multifusion m-category with $\Omega^{m-k}\mathcal{B} = k\mathrm{Vec}$, then $Z_{(k+1)}(\mathcal{A}) \to \mathcal{A}$ is dominant. For this, dimensionally reduce on blackboard-framed spheres to the E_1 case.

Classification of topological orders

Slogan: If all excitations of dimension $\leq k$ are "trivial," then all morphisms of codimension $\leq k+1$ are "trivial."

Example: Suppose $n = 3$. If (1) $\Omega^2\mathcal{A} = \mathrm{V{\scriptsize EC}}$ ("no lines") then (2) $\mathcal{A} = \Sigma^2\Omega^2\mathcal{A} = 3\mathrm{V{\scriptsize EC}}$.

This is the main unproven step in:

Theorem (Lan–Kong–Wen, Lan–Wen, JF): Each (3+1)D topological order is canonically an anomalous topological sigma models with target a 1-groupoid.

Small print: If $\mathcal{A}$ is fermionic, then target is the categorical spectrum $\mathrm{Spec}(\Omega^2\mathcal{A}) = \hom(\Omega^2\mathcal{A}, \mathrm{SV{\scriptsize EC}})$. Action is in reduced supercohomology (need anomalous/reduced to make canonical). If $\mathcal{A}$ is bosonic, then $\mathrm{Spec}(\Omega^2\mathcal{A})$ carries an action by $\mathbb{Z}_2^f[1]$, and action lives in reduced $\mathbb{Z}_2^f[1]$-twisted-equivariant supercohomology. Now can have actual anomaly, because twisted-equivariance means nonreduced $\neq$ reduced $\oplus \ldots$, but rather there is LES.

Classification in other dimensions?

(0+1)D Topological order = central simple algebra $\cong \mathrm{Mat}_N(\mathbb{C})$. N is the ground state degeneracy. Classification requires that $\mathbb{C}$ is algebraically closed. Otherwise, you could have a system which is protected by Galois symmetry.

(1+1)D All unstable, because $\mathcal{C}$ is algebraically closed. Fermionically, there is still some data: a relative Arf invariant between pairs of local vacua. This is explained by super cohomology.

(2+1)D Stable (aka fusion) topological orders = “MTCs” = nondegenerate braided fusion categories (no canonical ribbon structure). Classification of MTCs is wild.

Unstable (multifusion) topological orders: each local vacuum supports an MTC. Each pair of local vacua carries a Witt-equivalence of MTCs.

(3+1)D Anomalous topological sigma models with 1-groupoid targets.

(4+1)D I expect a classification in terms of symplectic finite abelian groups and Lagrangian correspondences. Joint work in progress with **Matthew Yu**.

(5+1)D Anomalous topological sigma models with 2-groupoid targets. Basically repeat the Lan–Kong–Wen proof. Need 2-categorical Tannakian duality: every symmetric separable multifusion 2-category admits a super fibre functor, i.e. a symmetric monoidal functor to to $2\mathrm{SV\small{EC}} := \Sigma\mathrm{SV\small{EC}} = \mathrm{SA\small{LG}}$. Joint work in progress with **Michael Hopkins**.

(6+1)D Probably something about the classical Witt group of finite abelian groups with nondegenerate quadratic forms (bosonic) or symmetric bilinear pairings (fermionic)? Definitely there are gravitational anomalies (7D Chern–Simons theory).

(7+1)D Would have classification in terms of anomalous topological sigma models with 3-groupoid targets, except 3-categorical Tannakian duality fails. Need beyond-fermionic 2-branes.

Thank you!

Further details:

[arXiv:1502.06526] (Op)lax natural transformations, twisted field theories, and the “even higher” Morita categories. (joint with Claudia Scheimbauer)

[arXiv:1905.09566] Condensations in higher categories. (joint with Davide Gaiotto)

[arXiv:2003.06663] On the classification of topological orders.

[these slides] http://categorified.net/AlgTopOrder.pdf