# Fermions and categorified algebra closure

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Based on joint work in progress with David Reutter

https://categorified.net/AustMS2025.pdf

# Why fermions?

Consider a quantum system with various species of (quasi)particles: quarks and leptons, phonons and Cooper pairs, . . . .

**Question:** Can you consistently model each species by a vector space of "internal states"? Requirements:

- Superposition of particles $\leftrightsquigarrow$ $\oplus$ of vector space.
- Fusion of particles $\leftrightsquigarrow$ $\otimes$ of vector spaces.
- Exchanging particles $\leftrightsquigarrow$ natural iso $\times : V \otimes W \cong W \otimes V$.

**In other words:** Does there necessarily exist a symmetric monoidal functor from the category of particle configurations to $\mathbf{Vec}$?

**Warning:** In ($\leq$2+1)D, particles can have nonsymmetric braiding statistics. Particle braiding is symmetric in ($\geq$3+1)D.

# Why fermions?

**Question:** Does there necessarily exist a symmetric monoidal functor from the category of particle configurations to $\mathbf{Vec}$?

**Answer:** No. For any vector space, the exchange isomorphism $: V \otimes V \cong V \otimes V$ has positive trace: it interferes constructively with the identity isomorphism. On the other hand, for physical protons or electrons, exchanging two identical particles interferes destructively with leaving them in place.

**Cheap fix (Grassmann, Dirac, Fermi):** Formally declare a new type of vector — a supervector — which comes in two parts. The even part satisfies the usual exchange statistics, but the odd part picks up an extra $-1$ under self-exchanges.

In other words, you use the category $\mathbf{sVec}$, which is almost the same as the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, except you add a $-1$ to the commutator.

# Why fermions?

**Cheap fix (Grassmann, Dirac, Fermi):** Formally declare a new type of vector — a supervector — which comes in two parts. The even part satisfies the usual exchange statistics, but the odd part picks up an extra $-1$ under exchanges.

**Theorem (Doplicher–Roberts, Deligne):** This cheap fix suffices! If $\mathcal{A}$ is any symmetric monoidal $\mathbb{C}$-linear category which is nonzero and not too large, then there exists a symmetric monoidal $\mathbb{C}$-linear functor $\mathcal{A} \to \mathbf{sVec}_{\mathbb{C}}$.

**Details:** For both Doplicher–Roberts and Deligne, not too large includes rigid: every object in $\mathcal{A}$ should have a dual object (e.g. dual of a finite-dimensional vector space). For Doplicher–Roberts, not too large = rigid + C*. For Deligne, not too large = rigid + finite-dimensional hom spaces + abelian + finite-length Jordan-Holder decompositions + $\mathrm{length}(X^{\otimes n}) \leq C^n$.

# Why imaginary numbers?

Consider a quantum system with various local integrals of motion: local operators whose value is locally constant as a function of their location in spacetime.

**Question:** Can you consistently model each integral of motion by a numerical "value"? Requirements:

- Superposition of operators $\leftrightsquigarrow$ $+$ of numbers.
- Fusion of operators $\leftrightsquigarrow$ $\times$ of numbers.

**In other words:** Does there necessarily exist a homomorphism from the commutative ring of integrals of motion to $\mathbb{R}$?

**Warning:** In ($\leq$0+1)D, integrals of motion can have noncommutative operator products. Operator product is commutative in ($\geq$1+1)D.

# Why imaginary numbers?

**Question:** Does there necessarily exist a homomorphism from the commutative ring of integrals of motion to $\mathbb{R}$?

**Answer:** No. The square of any real number is positive. But systems with spontaneously broken time-reversal symmetry have (real) integrals of motion that square to $-1$.

**Cheap fix (Cardano):** Formally declare a new type of number — a complex number — which comes in two parts. The real part satisfies the usual multiplication law, but the imaginary part picks up an extra $-1$ under self-multiplications.

In other words, you use the ring $\mathbb{C}$, which is almost the same as the ring of $\mathbb{Z}/2\mathbb{Z}$-graded real numbers, except you add a $-1$ to the multiplication law.

# Why imaginary numbers?

**Cheap fix (Cardano):** Formally declare a new type of number — a complex number — which comes in two parts. The real part satisfies the usual multiplication law, but the imaginary part picks up an extra $-1$ under self-multiplications.

**Theorem (Gauss, Hilbert, Gelfand):** This cheap fix suffices! If $A$ is any commutative $\mathbb{R}$-algebra which is nonzero and not too large, then there exists a commutative $\mathbb{R}$-algebra homomorphism $A \to \mathbb{C}$.

**Details:** For Gauss, not too large = singly-generated. For Hilbert, not too large = finitely-generated. For Gelfand, not too large = C*.

**Punchline:** $\mathbf{Vec}_{\mathbb{C}} \rightsquigarrow \mathbf{sVec}_{\mathbb{C}}$ is an algebraic closure!

$$\mathbb{R} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbb{C} \overset{\text{modules}}{\rightsquigarrow} \mathbf{Vec}_{\mathbb{C}} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbf{sVec}_{\mathbb{C}} \rightsquigarrow \dots$$

# Going deeper

$$\mathbb{R} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbb{C} \overset{\text{modules}}{\rightsquigarrow} \mathbf{Vec}_{\mathbb{C}} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbf{sVec}_{\mathbb{C}} \rightsquigarrow \dots$$

$n$-categories organize $n$-(spacetime-)dimensional objects in quantum systems: strings, defects, interfaces, branes, ....

**Question:** What kind of statistics will we need to introduce to realize all possible $n$-dimensional objects? What is the algebraically closed $n$-category? Does it even exist?

**Strategy:** Set $\mathcal{W}^0 := \mathbb{C}$, $\mathcal{W}^1 := \mathbf{sVec}_{\mathbb{C}}$, $\mathcal{W}^n :=$ algebraic closure of $\mathbf{Mod}(\mathcal{W}^{n-1})$. Build these inductively.

**Vocabulary advocacy:** Higher algebra has come to mean the algebra of homotopy types, spectra, $(\infty, 1)$-categories. $\mathbf{Vec}_{\mathbb{C}}$ has $\mathbb{C}$ at the top: sym mon $n$-categories feel like cohomological objects. So I advocate the term deeper algebra for the algebra of $(\infty, \infty)$-categories.

# Deeper roots of unity

**Lemma:** If $\mathbb{K}$ is an algebraically closed field of characteristic zero, then its roots of unity $\mu(\mathbb{K}) :=$ torsion subgroup of $\mathbb{K}^\times$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$. If $\mathbb{K}$ has characteristic $p$, then $\mu(\mathbb{K}) \cong \mathbb{Q}_p/\mathbb{Z}_p$.
**Proof:** Pick a finite group $A$ (of order coprime to $p$). Look at the group algebra $\mathbb{K}[A]$. There must be $|A|$-many maps $\mathbb{K}[A] \to \mathbb{K}$. But $\hom(\mathbb{K}[A], \mathbb{K}) = \hom(A, \mu(\mathbb{K}))$. This forces $\mu(\mathbb{K}) \cong \mathbb{Q}/\mathbb{Z}$.

Basically the same thing works in deeper categories, with some vocabulary change:

abelian group $\rightsquigarrow$ spectrum

finite abelian group $\rightsquigarrow$ $\pi$-finite spectrum

A spectrum is basically a coherently homotopy-commutative topological group. It is $\pi$-finite when the product of its homotopy groups is finite.

# Deeper roots of unity

The spectrum analogue of $\mathbb{Q}/\mathbb{Z}$ is called the Brown–Comenetz dual to spheres denoted $I_{\mathbb{Q}/\mathbb{Z}}$. Its homotopy groups:

$$\pi_n I_{\mathbb{Q}/\mathbb{Z}} = \begin{cases} 0, & n > 0 \\ \hom(\pi^s_{|n|}, \mathbb{Q}/\mathbb{Z}), & n \leq 0 \end{cases}$$

$\pi^s_m := \varinjlim_{k\to\infty} \pi_{m+k} S^k =$ stable homotopy groups of spheres.

**Theorem (Brown–Comenetz):** Consider the homotopy 1-category of the $\infty$-category of those spectra $A$ equipped with a map $\pi_0 A \to \mathbb{Q}/\mathbb{Z}$. This 1-category has a terminal object, $I_{\mathbb{Q}/\mathbb{Z}}$.

Proof is nontrivial. Existence requires $\mathbb{Q}/\mathbb{Z}$ is a divisible group.

# Deeper roots of unity

**Lemma:** If $\mathcal{W}^n$ is an algebraically closed symmetric monoidal $n$-category of characteristic zero, then its roots of unity $\mu(\mathcal{W}^n)$, i.e. the torsion subspectrum of $(\mathcal{W}^n)^\times$, is iso to $I^n_{\mathbb{Q}/\mathbb{Z}} := \Omega^\infty \Sigma^n I_{\mathbb{Q}/\mathbb{Z}}$.

**Main Theorem (JF–Reutter):** There is a unique sequence $\mathcal{W}^\bullet$ with:

1. $\mathcal{W}^n$ is a rigid sym mon $\mathbb{C}$-linear $n$-category with duals.
2. $\mathcal{W}^n$ is an ind. limit of finite separable extensions of $\mathbf{Mod}^n(\mathbb{C})$.
3. $\mathcal{W}^{n-1} \cong \mathrm{End}_{\mathcal{W}^n}(1)$, i.e. $\mathcal{W}^\bullet$ is a categorical spectrum.
4. $\mu(\mathcal{W}^n) \cong I^n_{\mathbb{Q}/\mathbb{Z}}$.

Moreover, $\mathcal{W}^n$ is separably closed among sym mon $n$-categories.

Item 2 can probably be weakened: when $n = 3$, there are extensions that feel inseparable but algebraic. Depending on the weakening, there are also transcendental extensions. Ask me in private.

# Deeper Kummer extensions

Suppose a field $\mathbb{K}$ has $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$. It probably is not algebraically closed. For example, it can still have noncyclotomic abelian extensions.

**Theorem (Kummer):** Suppose $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$ and $A$ is a finite abelian group with $A^\vee := \hom(A, \mathbb{Q}/\mathbb{Z})$. Then $A$-Galois extensions of $\mathbb{K}$ are classified by $\mathrm{Ext}^1(A^\vee, \mathbb{K}^\times) = \hom_{\mathbf{Sp}}(A^\vee, (\mathbf{Vec}_{\mathbb{K}})^\times)$.

**Proof:** An $A$-Galois extension $\mathbb{K} \to \mathbb{L}$ is in particular an $A$-representation. If $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$, then every $A$-representation is diagonalizable into one-dimensional eigenspaces. This diagonalization writes $\mathbb{L} = \bigoplus_{\alpha \in A^\vee} \mathbb{L}_\alpha$ as an $A^\vee$-graded commutative algebra. The assignment $\alpha \mapsto \mathbb{L}_\alpha$ is a symmetric monoidal functor $A^\vee \to \mathbf{Vec}_{\mathbb{K}}$, or equivalently a spectrum map $A^\vee \to (\mathbf{Vec}_{\mathbb{K}})^\times$.

# Deeper Kummer extensions

**Theorem (Kummer):** Suppose $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$ and $A$ is a finite abelian group with $A^\vee := \hom(A, \mathbb{Q}/\mathbb{Z})$. Then $A$-Galois extensions of $\mathbb{K}$ are classified by $\mathrm{Ext}^1(A^\vee, \mathbb{K}^\times) = \hom_{\mathbf{Sp}}(A^\vee, (\mathbf{Vec}_{\mathbb{K}})^\times)$.

Since $A^\vee$ is ($\pi$-)finite, any map to $(\mathbf{Vec}_{\mathbb{K}})^\times$ factors through its torsion subgroup $\mu(\mathbf{Vec}_{\mathbb{K}})$.

**Punchline:** Abelian extensions of $\mathbb{K}$ are controlled by the roots of unity in $\mathbf{Vec}_{\mathbb{K}}$.

A field without abelian extensions is solvably closed. To form the solvable closure, you keep adjoining abelian extensions until $\mu(\mathbf{Vec}_{\mathbb{K}})$ is as simple as possible.

Nonabelian simple groups are much harder. Fortunately they won't arise for us.

# Deeper Kummer extensions

Using some long exact sequences, we showed:

**Theorem (JF–Reutter):** Suppose $\mathcal{W}^{n-1}$ is algebraically closed, in particular $\mu(\mathcal{W}^{n-1}) = I^{n-1}_{\mathbb{Q}/\mathbb{Z}}$. Then $\mathbf{Mod}(\mathcal{W}^{n-1})$ has a universal abelian extension: there is a universal abelian group $U^n$ such that

$$\pi_0\{A\text{-graded extensions of } \mathbf{Mod}(\mathcal{W}^{n-1})\} = \hom_{\mathbf{AbGp}}(A, U^n).$$

Moreover, this universal $U^n$-extension (classified by $\mathrm{id} : U^n \to U^n$) is algebraically closed. It is $\mathcal{W}^n$.

The dual group $(U^n)^\vee = \hom(U^n, \mathbb{Q}/\mathbb{Z})$, or rather some cohomological shift of it, is the relative Galois group $\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1})$. The full Galois group $\mathbf{Gal}(\mathcal{W}^n/\mathbb{C})$ is a homotopical group with $\pi_n\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1}) = (U^n)^\vee$.

# Why don't we see nonabelian extensions?

Our proof is direct and algebraic. Indeed, a priori there is not a well-defined algebraic closure. We need to construct it without knowing its existence, in particular without a general deeper Galois theory. But what seems to be going on, with hindsight, is:

- If everything works well, then definitely $\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1}) = K(\pi_n \mathbf{Gal}(\mathcal{W}^n/\mathbb{C}), n)$ is abelian.
- Any homotopical group with trivial $\pi_0$ is solvable.
- Given a categorical spectrum $\mathcal{A}^\bullet$, its roots of unity $\mu(\mathcal{A}^\bullet)$ are some sort of "étale cohomology" of "$\mathrm{Spec}(\mathcal{A}^\bullet)$." Saying $\mu(\mathcal{W}^\bullet) = I_{\mathbb{Q}/\mathbb{Z}}$ is saying that $\mathrm{Spec}(\mathcal{W}^\bullet)$ is a homology point.
- On the other hand, the ordinary Galois group $\mathrm{Gal}(\overline{\mathcal{A}^0}/\mathcal{A}^0)$ measures "étale $\pi_1$." Saying $\mathcal{W}^0 = \mathbb{C}$ is saying that $\mathrm{Spec}(\mathcal{W}^\bullet)$ has the fundamental group of a point.
- Whitehead, Hurewicz: A space with both the homology and fundamental group of a point is a point.

# Calculating the Galois group

An ordinary absolute Galois group has a cyclotomic character $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^{\times}$. Our cyclotomic character lands in $\mathrm{Aut}(I_{\mathbb{Q}/\mathbb{Z}}) = \widehat{\mathbb{S}}^{\times}$.

Using ideas from (un)gauging in TQFTs, from abelian Chern–Simons theory, and from surgery theory, we can show:

**Theorem (JF–Reutter):** The fibre of the cyclotomic character admits an invariant valued in the L-theory of $\pi$-finite spectra. This invariant an isomorphism in degrees $\geq 5$.

Calculating that L-theory, we find a LES ($n \geq 5$):

$$\cdots \to \pi_{-n} I_{\mathbb{Q}/\mathbb{Z}} \to U^n \to \left\{ \begin{array}{ll} 0, & n \text{ even} \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \pmod 4 \\ \text{Witt group}, & n \equiv 3 \pmod 4 \end{array} \right\} \to \dots$$

# Physics application

By analyzing our LES, and assuming that {gapped topological phases of matter} assemble into a nice deeper category, we can show:

**Physics Theorem (JF–Reutter):** Suppose $I$ is an invertible (=short range entangled) phase of gapped topological matter. If $I$ admits a gapped topological boundary condition, then the partition function of $I$ on stably-framed manifolds is either trivial or an Arf–Brown–Kervaire invariant. If the partition function of $I$ is any other invariant of framed manifolds, then $I$ has enforced gaplessness (=conductivity) on its boundary.

# Credit where it's due

Idea that $\mathbf{Vec}_{\mathbb{C}} \to \mathbf{sVec}_{\mathbb{C}}$ is an algebraic closure: my "Spin, Statistics" paper, inspired by jt work with Chirvasitu.

Idea to look for a categorical spectrum with $\mathcal{W}^1 = \mathbf{sVec}_{\mathbb{C}}$ and $\mu(\mathcal{W}^\bullet) = I^\bullet_{\mathbb{Q}/\mathbb{Z}}$: Hopkins, Freed–Hopkins.

$\mathbf{Mod}(\mathbf{sVec})$ is algebraically closed: conjectured by Hopkins; immediate consequence of my "grouplike" theorem jt with Yu.

Constructing $\mathcal{W}^3$: Freed–Scheimbauer–Teleman. They told us their answer, and we back-solved to find the right question.

Surgery analysis of Galois group: inspired by Lan–Kong–Wen classification of 4D topological orders.

Narrative in terms of Kummer extensions: inspired by Burklund–Schlank–Yuan work on chromatic homotopy theory.

# THANK YOU

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