Applications of BRST gauge fixing: Chern–Gauss–Bonnet and the volume of $X//\mathrm{T}X$

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Contents

1 Review: Q-manifolds and Lie algebroid measures

We begin by recalling some of what was said last week.

Definition: *We denote by* $\mathbb{R}^\times$ *(resp.* $\mathbb{C}^\times$*) the group of nonzero real (complex) numbers under multiplication. A* $\mathbb{Z}$-graded supermanifold *is a supermanifold* X *with a distinguished (left, say)* $\mathbb{R}^\times$ *action, such that the corresponding action on* $\mathscr{C}^\infty(X) \otimes \mathbb{C}$ *extends to a holomorphic action by* $\mathbb{C}^\times$*. Given a vector bundle* $A \to X$*, we let* πA *denote the parity-reverse total space with the usual* $\mathbb{R}^\times$*-action on the fibers, and* $\pi^{-1}A$ *the parity-reverse vector bundle with the opposite* $\mathbb{Z}$*-grading. We will call the distinguished* $\mathbb{Z}$*-grading the "cohomological degree."*

A Q-manifold *is a supermanifold with a distinguished action by* $\mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$*, such that the* $\mathbb{R}^\times$ *action makes the supermanifold* $\mathbb{Z}$*-graded. We will call the differential of the* $\mathbb{R}^{0|1}$ *action the* Q-structure *or* cohomological vector field*, and denote it by* Q*. The letter "Q" is for "cohomology."*

Remark: The reason for asking that the $\mathbb{R}^\times$ action extend to a $\mathbb{C}^\times$ action is to assure that the eigenvalues for the infinitesimal generator of this action be integers. If we were thinking more algebraically, we would have used the affine algebraic group scheme $\mathbb{G}_m$, which is defined via Tannaka duality as the group whose representation theory over a commutative ring R consists of $\mathbb{Z}$-graded R-modules.

In all our examples, the $\mathbb{Z}$-grading will agree (mod 2) with the fermionic ($\mathbb{Z}/2$) grading. If the $\mathbb{Z}$- and fermionic gradings agree, then the algebra of functions on a Q-manifold is a commutative dga. Allowing the $\mathbb{Z}$- and fermionic gradings to disagree is equivalent to working in the category of commutative dg super algebras (chain complexes of super vector spaces).

It is common also to ask that the supermanifold contain local charts in which the coordinate functions are homogeneous for the $\mathbb{R}^\times$ action. $\diamondsuit$

Q-manifolds are a (somewhat rudimentary) language for "derived geometry." For example, there are natural notions of the *homology* of a Q-manifold, whether a map of Q-manifolds is a *quasi-isomorphism*, and what a *homotopy equivalence* of Q-manifolds is. Our interest in Q-manifolds will be that they provide models of *derived stacks*.

Definition: *A* Lie algebroid *over X consists of a vector bundle $A \to X$, a vector bundle map $\rho : A \to \mathrm{T}X$, and the structure on $\Gamma(A)$ of a sheaf of Lie algebras (over $\mathbb{R}$, but not usually over $\mathscr{C}^\infty(X)$), such that the Leibniz rule $[a, fb] = f[a, b] + \rho(a)[f]\, b$ holds for $a, b \in \Gamma(A)$, $f \in \mathscr{C}^\infty$. It follows that $\rho : \Gamma(A) \to \Gamma(\mathrm{T}X)$ is a map of (sheaves of) Lie algebras. Lie algebroids are an infinitesimal version of Lie groupoids.*

Let $A \to X$ be a Lie algebroid. A representation *of A is a quasicoherent sheaf over X which is simultaneously a sheaf of $\mathscr{C}^\infty$-modules and a sheaf of $\Gamma(A)$-modules, and such that a similar Leibniz rule is satisfied. For example, $\mathscr{C}^\infty$ is an A-module, whereas $\Gamma(A)$ is not. The category of A-modules is abelian. The* derived classifying space *or* derived quotient *$X//A$ is cdga $\mathrm{Rhom}_A(\mathscr{C}^\infty, \mathscr{C}^\infty)$, where* Rhom *is the right-derived functor of* hom *(the one that computes* $\mathrm{Ext}^\bullet$*) — it is defined only up to homotopy.*

For any Lie algebroid $A \to X$, there is a canonically-defined Q-manifold structure on the parity-reversed total space πA which presents the derived quotient $X//A$.

Example (Main example): The tangent bundle $\mathrm{T}X \to X$ is itself a Lie algebroid. The canonical presentation of $X//\mathrm{T}X$ is $X_{\mathrm{dR}} \overset{\mathrm{def}}{=} \mathrm{spec}(\Omega^\bullet(X)) = \pi\mathrm{T}X$, with the Q-structure encoding the de Rham d. $\diamondsuit$

The canonically defined Q-manifold presenting a given derived quotient is ill-suited for certain constructions. Fortunately, we have at our disposal many other presentations of the same derived quotient:

Theorem (—, DBE): *Let $Y \to X$ be any graded vector bundle, and $A \to X$ a Lie algebroid (or more generally replace πA by any Q-manifold and $\pi A \to X$ any map of graded manifolds). The pullback in Q-manifolds $Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \pi A$ presents the derived quotient $X//A$ (it is homotopy equivalent to the original Q-manifold).*

Remark: This pullback is studied by Gacía-Saz and Mehta as an example of a "Lie algebroid vector bundle"; it is in their language "type 1." The underlying graded manifold of $Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \pi A$ is isomorphic to $Y \times_X \pi Y \times_X \pi A$, but choosing such an isomorphism amounts to choosing a connection on a certain bundle, and so the isomorphism is not canonical. $\diamondsuit$

Definition (Weinstein): *Let $A \to X$ be a Lie algebroid. Neither A nor $\mathrm{T}X$, nor in fact $A^{\wedge \mathrm{rank}}$ nor $\mathrm{T}^{\wedge \dim} X$, are themselves A-modules. Nevertheless, the ratio of lines $(\mathrm{T}^*)^{\wedge \dim} \otimes A^{\wedge \mathrm{rank}}$ is an A-module in a canonical way. At least when everything is oriented, a* Lie algebroid measure *on $A \to X$ is an A-invariant section of $(\mathrm{T}^*)^{\wedge \dim} \otimes A^{\wedge \mathrm{rank}}$. One can think of such a section as a "ratio": a measure on X divided by a measure on the fibers of A.*

Remark: It would be interesting to investigate the derived space $\mathrm{Rhom}\big(\mathscr{C}^\infty, (\mathrm{T}^*)^{\wedge \dim} \otimes A^{\wedge \mathrm{rank}}\big)$, and see to what extent our constructions generalize. We have not done so. ◊

Example: The de Rham algebroid $\mathrm{T}X \to X$ has a canonical measure (if everything is oriented).◊

Let $A \to X$ be a Lie algebroid. A choice of Lie algebroid measure on A determines a Berezinian measure on πA which is Q-invariant. It is not invariant under the $\mathbb{R}^\times$ action; rather, it is in cohomological degree $-\operatorname{rank} A$.

More generally, the Lie algebroid measure determines a Berezinian measure on any presentation constructed in the theorem above, by multiplying the Berezinian measure on πA with the canonical measure on the fibers of $Y_{\mathrm{dR}} \to X_{\mathrm{dR}}$. The cohomological degree of this measure on $Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \pi A$ depends on the graded rank of Y. If Y is locally isomorphic to $\pi^{-1}A$, then $Y_{\mathrm{dR}} \times_{X_{\mathrm{dR}}} \pi A$ receives a Berezinian measure in cohomological degree 0. We remark that, since we are working over $\mathbb{R}$, there exist non-canonical isomorphisms $\pi^{-1}A \cong \pi^{-1}A^* = (\pi A)^*$.

2 $\mathrm{Vol}(X//\mathrm{T}X)$ and one side of the Chern–Gauss–Bonnet theorem

In this section we study in detail the canonical Lie algebroid measure on $\mathrm{T}X \to X$. We suppose for convenience that X is oriented. Our goal will be to compute the integral of 1 against this measure (over the derived stack $X//\mathrm{T}X$). We will see that the correct answer is the Euler characteristic $\chi(X)$. Along the way we will prove (part of) the Chern–Gauss–Bonnet theorem, equating the integral over X of scalar curvature (for some metric) with the signed count of critical points of a Morse function.

As we have already said, the Lie algebroid $\mathrm{T}X \to X$ has a canonical algebroid measure. When X is oriented, the corresponding measure on $\pi\mathrm{T}X = \mathrm{spec}(\Omega^\bullet(X))$ is the "integrate top forms" measure. We instead choose a presentation of the derived stack $X//\mathrm{T}X$ for which the Berezinian measure corresponding to the canonical algebroid measure is in cohomological degree 0. Our choice will be to take $Y = \pi^{-1}\mathrm{T}X$ in the above theorem.

We choose local coordinates x^i on X. Then the corresponding fiber coordinates on $\pi\mathrm{T}X$ we denote by ξ^i — under changes of coordinates $x \rightsquigarrow \tilde{x}(x)$, the ξs transform as $\tilde{\xi}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\xi^j$. On $Y = \pi^{-1}\mathrm{T}X$ we choose fiber coordinates $\bar{\xi}$, transforming similarly to the ξs. The Lie algebroid pullback is $Y_{\mathrm{dR}} = \pi\mathrm{T}(\pi^{-1}\mathrm{T}X)$. Fiber coordinates of this corresponding to the xs we continue to denote by ξ, and corresponding to $\bar{\xi}$ we denote by b^i. Then the b^i do not transform as tensors, but rather under $x \rightsquigarrow \tilde{x}$ they transform via:

$$\tilde{b}^i = \frac{\partial \tilde{x}^i}{\partial x^j} b^j + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \xi^j \bar{\xi}^k$$

or maybe I'm off by a sign. We mention this just to correct something we have often got wrong. There is an isomorphism $\pi\mathrm{T}(\pi^{-1}\mathrm{T}X) \cong (\pi\mathrm{T} \oplus \mathrm{T} \oplus \pi^{-1}\mathrm{T})X$, but choosing such an isomorphism requires choosing a connection.

In any case, the Q-structure on $\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)$ is just the de Rham d:

$$Q = b^i \frac{\partial}{\partial \bar\xi^i} + \xi^i \frac{\partial}{\partial x^i}$$

The berezinian measure is

$$\mu = \mathrm{d}x^1 \cdots \mathrm{d}x^{\dim} \, \mathrm{d}\xi^1 \cdots \mathrm{d}\xi^{\dim} \, \mathrm{d}\bar\xi^1 \cdots \mathrm{d}\bar\xi^{\dim} \, \mathrm{d}b^1 \cdots \mathrm{d}b^{\dim}$$

up to a sign convention.

Our goal is now to calculate, or rather to define, the value of

$$\int_{\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)} \mu.$$

We immediately run into a problem. The integral does not converge absolutely. Rather, if we integrate out the fermionic directions, we get zero, whereas the integral in the b direction is infinite. The BRST argument provides a way to modify the integral into a convergent one, by multiplying the integrand by $\exp(Q[f])$ for some $f \in \mathscr{C}^\infty(\pi\mathrm{T}(\pi^{-1}\mathrm{T}X))$ in cohomological degree -1. With luck, $\int \exp(Q[f])\mu$ will converge absolutely. By the BRST argument, provided it converges sufficiently quickly, the value of $\int \exp(Q[f])\mu$ is independent of f. (We use the fact that "sufficiently quickly" is a convex condition in f.)

Where can we find functions in cohomological degree -1? The supermanifold $\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)$ comes equipped with another piece of structure that we have not discussed. Recall that there is a canonical isomorphism $\pi\mathrm{T}(\pi^{-1}\mathrm{T}X) = \mathrm{Maps}(\pi^{-1}\mathbb{R} \times \pi\mathbb{R}, X)$, and that the de Rham cohomological vector field Q can be thought of as the translation action in the $\pi^{-1}\mathbb{R}$ direction. From this perspective it is then clear that there is another vector field $\bar Q$ corresponding to translation in the $\pi\mathbb{R}$ direction. It commutes with itself and with Q, and is in cohomological degree -1. In local coordinates,

$$\bar Q = -b^i \frac{\partial}{\partial \xi^i} + \bar\xi^i \frac{\partial}{\partial x^i}$$

Thus one source of functions in cohomological degree -1 is those functions of the form $\bar Q[f]$ for $|f| = 0$.

For example, choose a metric $g_{ij}(x)$ on X. We claim that $f(x, \xi, \bar\xi, b) = \frac{1}{2} g_{ij}(x)\, \xi^i \bar\xi^j$ is a globally-defined coordinate-independent function. Indeed, writing $(\theta, \bar\theta)$ for the coordinate functions on $\pi^{-1}\mathbb{R} \times \pi\mathbb{R}$ and $\mathbf{x}(\theta, \bar\theta)$ for a generic map $\pi^{-1}\mathbb{R} \times \pi\mathbb{R} \to X$, we see that $f(\mathbf{x}) = \frac{1}{2}\langle \frac{\partial \mathbf{x}}{\partial \theta}, \frac{\partial \mathbf{x}}{\partial \bar\theta} \rangle_g$, where the partial derivatives are evaluated at $\theta = 0 = \bar\theta$.

Then the BRST argument suggests that, modulo convergence issues,

$$\int_{\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)} \mu = \int_{\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)} \exp\big(Q\bar Q f\big)\, \mu.$$

What the BRST argument actually proves is that, provided the right-hand side converges absolutely, the value of the right-hand side is independent of f. The function $Q\bar{Q}f$ is called the *gauge-fixed action* s_{GF}.

So we calculate:

$$s_{\mathrm{GF}} = \frac{1}{2}Q\bar{Q}\big[g_{ij}(x)\,\xi^i\bar{\xi}^j\big] = \frac{1}{2}Q\big[-g_{ij}\,b^i\bar{\xi}^j + \partial_k g_{ij}\,\bar{\xi}^k\xi^i\bar{\xi}^j\big] =$$
$$= -\frac{1}{2}\,g_{ij}\,b^ib^j - \frac{1}{2}\,\partial_k g_{ij}\,\xi^k b^i\bar{\xi}^j + \frac{1}{2}\,\partial_k g_{ij}\,b^k\xi^i\bar{\xi}^j + \frac{1}{2}\,\partial_k g_{ij}\,\bar{\xi}^k\xi^i b^j + \frac{1}{2}\,\partial_l\partial_k g_{ij}\,\xi^l\bar{\xi}^k\xi^i\bar{\xi}^j =$$
$$= -\frac{1}{2}\,g_{ij}\,b^ib^j - \Gamma_{ijk}\,\xi^i\bar{\xi}^j b^k + \frac{1}{2}\,\partial_l\partial_k g_{ij}\,\xi^l\bar{\xi}^k\xi^i\bar{\xi}^j$$

where $\Gamma_{ijk} = \frac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\big)$ is the Christoffel symbol for the metric g in the chosen coordinates.

We can now integrate $\int \exp(s_{\mathrm{GF}})\mu$ as it converges absolutely, provided g is positive definite. Indeed, the b coordinate appears as a pure Gaussian with negative-definition quadratic part, and so we can integrate it out:

$$\int \exp(s_{\mathrm{GF}})\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{d}b =$$
$$= (2\pi)^{\dim X/2}\int \exp\left(\frac{1}{2}g^{kk'}\,\Gamma_{ijk}\xi^i\bar{\xi}^j\,\Gamma_{i'j'k'}\xi^{i'}\bar{\xi}^{j'} + \frac{1}{2}\,\partial_l\partial_k g_{ij}\,\xi^l\bar{\xi}^k\xi^i\bar{\xi}^j\right)\,\frac{1}{\sqrt{g}}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}$$

We can recognize the exponent as

$$\frac{1}{2}g^{kk'}\,\Gamma_{ijk}\xi^i\bar{\xi}^j\,\Gamma_{i'j'k'}\xi^{i'}\bar{\xi}^{j'} + \frac{1}{2}\,\partial_l\partial_k g_{ij}\,\xi^l\bar{\xi}^k\xi^i\bar{\xi}^j = -\frac{1}{4}R_{ijkl}(x)\,\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l$$

where $R_{ijkl}(x)$ is the Riemann curvature tensor.

Finally, one recalls the definition of fermionic integration in terms of Pfaffians and what not, and computes the integral, taking more care than we have been with signs. One gets:

$$\int_{\pi\mathrm{T}(\pi^{-i}\mathrm{T}X)} \exp(s_{\mathrm{GF}})\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}\,\mathrm{d}b = \dots = (2\pi)^{\dim X/2}(2\pi)^{-\dim X}\int_X (\text{a certain Pfaffian})\,\sqrt{g}\,\mathrm{d}x$$

If we know the Chern–Gauss–Bonnet theorem, we can immediately recognize the right-hand side as an expression for the Euler characteristic $\chi(X)$. For details, see the thesis by Dan Berwick-Evans.

3 The other side of the theorem, generalizations, and remarks

Here's an entirely different way to gauge-fix the integral $\int_{\pi\mathrm{T}(\pi^{-1}\mathrm{T}X)}\mu$, which is much closer to the original Faddeev-Popov approach. We want to write down a gauge-fixed action $s_{\mathrm{GF}} = Q[f]$. We choose a 1-form $\alpha_i(x)\,\mathrm{d}x^i$ on X, and set $f = \alpha_i\bar{\xi}^i$.

(Before we go further, an important remark is in order. When the 1-form $\alpha = \mathrm{d}h$ is exact, then $\alpha_i \xi^i = \bar{Q}[h(x)]$, so that this choice fits into our above rubric of finding degree-(-1) functions as $\bar{Q}$(degree-0 functions).)

Then:

$$s_{\mathrm{GF}} = Q\big[\alpha_i(x)\,\bar{\xi}^i\big] = \alpha_i\, b^i + \partial_i\alpha_j\,\xi^i\bar{\xi}^j$$

What happens when we integrate out the b variables? If α is real, then $\int \exp(\alpha_i\, b^i)$ diverges horribly. But if we switch $\alpha \mapsto \sqrt{-1}\alpha$ to make it pure imaginary, then we have

$$\int \exp\left(\sqrt{-1}\,\alpha_i\, b^i\right) \mathrm{d}b = (2\pi)^{-\dim X}\delta(\alpha_i).$$

Suppose that α has isolated nondegenerate zeros. Then we can integrating out the ξs at the same time, recognizing that in combination the integral is a Gaussian. The integral gives a determinant, and all together we get

$$\mathrm{Vol} = (2\pi)^{-\dim X}(2\pi)^{\dim X}\int_X \delta(\alpha_i)\,\det(\sqrt{-1}\partial_i\alpha_j)\,\mathrm{d}x.$$

But this is just a signed count of zeros of α, up to an overall $\sqrt{-1}^{\dim X}$. The sign is precisely the sign of the determinant of $\partial_i\alpha_j$, recalling that $\delta(ax) = \frac{1}{|a|}\delta(x)$ is positive. Thus we have shown that an integral of the Pfaffian of curvature of a metric is equal to a signed count of the zeroes of a 1-form. More care must be taken with the overall constants and so on, but these only depend on the dimension of X, and can be checked by checking the result on some particular nontrivial manifold.

One can also consider the situation when the zero locus of α does not consist of isolated nondegenerate points, but rather of an embedded submanifold of X. If $\partial\alpha$ is nondegenerate in the directions transverse to the submanifold, then similar arguments also allow to show that the Euler characteristic of X can be computed in terms of (signed) Euler characteristics of the connected components of the zero locus of α. For example, we might have $\alpha = \partial h$ where h is a Morse-Bott function.

Remark: You should probably be a bit concerned by the conditional convergence — the BRST argument is only really correct when convergence of the integrals is sufficiently strong. A more honest thing to do is to consider the sum $Q[\alpha_i\bar{\xi}^i + \bar{Q}[\frac{1}{2}g_{ij}\xi^i\bar{\xi}^j]]$. Then the integral in b converges absolutely. Integrating out this part gives

$$(2\pi)^{\dim/2}\int \exp\left(-\frac{1}{4}R_{ijkl}\,\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l + \partial_i\alpha_j\,\xi^i\bar{\xi}^j - g^{kl}\,\Gamma_{ijk}\xi^i\bar{\xi}^j\,\alpha_l + \frac{1}{2}g^{ij}\,\alpha_i\,\alpha_j\right)\frac{1}{\sqrt{g}}\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}$$

We can then consider rescaling the family of gauge-fixed actions $Q[\lambda\alpha_i\bar{\xi}^i + \bar{Q}[\frac{1}{2}g_{ij}\xi^i\bar{\xi}^j]]$, where λ is some parameter. In the limit as $\lambda \to \infty$, the integral continues to localize on the zeros of α. ♢

Remark: We will conclude this section by making a few more remarks about the operator $Q\bar{Q} = -\bar{Q}Q$. It is a composition of commuting vector fields, and so we will call it the *BRST Laplacian* (it has nothing to do with the *BV Laplacian* that you might have heard about, but no matter). See, in both computations, at least when α was exact, our gauge-fixed action wasn't just $s_{\mathrm{GF}} = Q[\text{something}]$, but in fact it was $Q\bar{Q}[\text{something}]$. We will relate this to path integrals.

Recall that we are integrating over the space $\pi\mathrm{T}(\pi^{-1}\mathrm{T}X) = \mathrm{Maps}(\pi^{-1}\mathbb{R} \times \pi\mathbb{R}, X)$. Suppose we have $f : \mathrm{Maps} \to \mathbb{R}$. We will begin by describing very explicitly the function $Q\bar{Q}f$.

There is a right action by translation $(\pi^{-1}\mathbb{R} \times \pi\mathbb{R}) \times (\pi^{-1}\mathbb{R} \times \pi\mathbb{R}) \to \pi^{-1}\mathbb{R} \times \pi\mathbb{R}$, and this induces a left action $(\pi^{-1}\mathbb{R} \times \pi\mathbb{R}) \times \mathrm{Maps} \to \mathrm{Maps}$. Precomposing, we get $f : (\pi^{-1}\mathbb{R} \times \pi\mathbb{R}) \times \mathrm{Maps} \to \mathbb{R}$. Fix $\mathbf{x} \in \mathrm{Maps}(\pi^{-1}\mathbb{R} \times \pi\mathbb{R}, X)$, and consider differentiation $f(-, \mathbf{x})$ in the $(\pi^{-1}\mathbb{R} \times \pi\mathbb{R})$ variables. Denoting these variables by $\theta, \bar{\theta}$, one sees that $(Q\bar{Q}f)(\mathbf{x}) = \frac{\partial^2}{\partial\theta\partial\bar{\theta}} f(\theta, \bar{\theta}, \mathbf{x})\Big|_{(\theta,\bar{\theta})=0}$.

But now let $f(\theta)$ be any function of an odd variable θ. It is a general fact about fermionic measures that $\int f(\theta)\,\mathrm{d}\theta = \frac{\partial f}{\partial \theta}\Big|_{\theta=0}$, up to a sign convention — this is how fermionic measures are defined.

So what we see is the following. To the function $f : \mathrm{Maps} \to \mathbb{R}$ we have associated a *Lagrangian density* for a field theory on $\pi^{-1}\mathbb{R} \times \pi\mathbb{R}$: to a path $\mathbf{x}$, the density at position $(\theta, \bar{\theta})$ is $f(\theta, \bar{\theta}, \mathbf{x})\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta}$. Then the gauge-fixed action $Q\bar{Q}f$ is precisely $\int_{\pi^{-1}\mathbb{R}\times\pi\mathbb{R}} f(\theta, \bar{\theta}, \mathbf{x})\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta}$, the action for the corresponding Lagrangian density. Conversely, to a Lagrangian density $f(\theta, \bar{\theta}, \mathbf{x})$ the corresponding gauge fixing condition is $f(0, 0, \mathbf{x})$. All together, we have constructed a correspondence between, on the one hand, ("time" independent) Lagrangians for a path integral over $\pi^{-1}\mathbb{R} \times \pi\mathbb{R}$ whose paths are valued in X, and, on the other hand, gauge-fixings of the trivial action on $\pi\mathrm{T}X$ of the form $s_{\mathrm{GF}} = Q\bar{Q}[\text{something}]$. ◊