Based on joint work with Owen Gwilliam and on ongoing conversations with Shamil Shakirov.

0. Introduction

Three questions to ask about integrals:

1 (Calculus). Given algebraic description of measure μ, compute $\langle f \rangle = \int f \mu$ for various functions f.

2 (Inverse problem). Given a priori knowledge of "shape" of measure μ, and some values of $\langle f \rangle$, compute μ.

3 (Physics). Given a priori "shape" of μ and some values of $\langle f \rangle$, compute more values of $\langle f \rangle$.

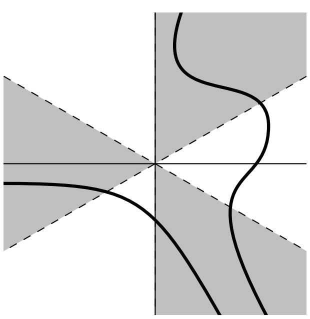
Ur-example: Wick's Thm (Isserlis, 1918): Let $\mu = e^s \mathrm{dLeb}$, with dLeb=Lebesgue, and $s(x) = \sum_{ij} s_{ij} \frac{x_i x_j}{2}$ with $\Re(s_{ij})$ negative-definite. Then:

1. $\int_{\mathbb{R}^n} f\mu = \frac{1}{\sqrt{\det(-2\pi s)}} \exp\left(\sum_{ij} \frac{(s^{-1})_{ij}}{2} \frac{\partial^2}{\partial x_i \partial x_j}\right) f$
2. $(s^{-1})_{ij} = \langle x_i x_j \rangle / \langle 1 \rangle$.
3. $\langle f \rangle$ is determined by $\langle 1 \rangle$ and $\langle x_i x_j \rangle$ just for variables appearing in f. If s inhomogeneous, include $\langle x_i \rangle$.

Question 3 lets us predict experiments, and is most important, but also hard. I will focus on question 1.

1. The problem, contours, and Stokes' theorem

Decide to study integrals of form $\int_\gamma f e^s \mathrm{dLeb}$ for $s : \mathbb{C}^n \to \mathbb{C}$ degree d, and $f \in \mathbb{C}[x_1, \dots, x_n]$. Must choose a *contour* γ — n-real-dimensional submanifold of $\mathbb{C}^n$. For convergence, want $\Re(s) \to -\infty$ at ends of γ.



Two possible contours for $n = 1$, $s(x) = x^3$.

Stokes' Theorem part 1: $\int_\gamma$ depends only on class of γ in relative homology group $\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) \ll 0\})$.

If highest-order part $s^{(d)}$ of s is smooth, then $\{\Re(s) \ll 0\} \simeq \{\Re(s^{(d)}) < 0\}$. Long exact sequence $\Rightarrow$

$$\mathrm{H}_n(\mathbb{C}^n, \{\Re(s) \ll 0\}) = \frac{\mathrm{H}_{n-1}(\{\Re(s) \ll 0\})}{\mathrm{H}_{n-1}(\mathbb{C}^n)}.$$

E.g. above two contours are a basis.

Stokes' Theorem part 2: $\int f e^s$ depends only on class of f in $\mathbb{C}[x_1, \dots, x_n]/(\text{image of maps } f \mapsto \frac{\partial f}{\partial x_i} + \frac{\partial s}{\partial x_i} f)$.

Cor: We have pairing

$$\mathrm{H}_n\big(\mathbb{C}^n, \{\Re(s) \ll 0\}\big) \otimes \frac{\mathbb{C}[x_1, \dots, x_n]}{\sum_i \text{image of } \frac{\partial}{\partial x_i} + \frac{\partial s}{\partial x_i}.} \to \mathbb{C}$$

Conj (easy? hard? false?): This pairing is perfect.

H_n is an object of topology and real algebraic geometry, and is inaccessible to pure algebra. $\mathbb{C}[x]/()$ is an object of pure algebra. For remainder of talk, pure algebra.

2. Turning the problem into homological algebra

Important problem-solving technique: resolve quotients into chain complexes.

Build graded-com algebra $V_\bullet = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ with $|\xi_i| = +1$ in homological degree (so anticommute). Make into complex *but not CDGA* with differential

$$\partial_{\text{full}} = \underbrace{\sum_i \frac{\partial s}{\partial x_i} \frac{\partial}{\partial \xi_i}}_{\partial_{\text{cl}}} + \underbrace{\sum_i \frac{\partial^2}{\partial x_i \partial \xi_i}}_{\partial_{\text{Leb}}}$$

Geometric interpretation: $V_\bullet =$ antisymmetric multivector fields. $\partial_{\text{Leb}} =$ divergence. $\partial_{\text{cl}} =$ contract with $\mathrm{d}s$.

Supergeometric interpretation: $V_\bullet = \mathcal{O}(\Pi \mathrm{T}^* \mathbb{C}^n)$. Principal symbol of ∂_{full} is canonical Poisson bivector.

Important problem-solving technique: Study chain complexes by breaking differential into simpler pieces.

Homotopy perturbation lemma (1960s):
Suppose given a *retraction* (in any additive category)

$$(H_\bullet, \partial_H) \underset{\phi}{\overset{\iota}{\leftrightarrows}} (V_\bullet, \partial) \circlearrowright \eta \qquad \begin{array}{l} \iota\phi = \mathrm{id}_H \\ \phi\iota = \mathrm{id}_V - [\partial, \eta] \end{array}$$

and a *perturbation* $\partial \rightsquigarrow \partial + \delta$ with $(\partial + \delta)^2 = 0$. *If* $(\mathrm{id}_V - \delta\eta)$ *is invertible*, get new retraction:

$$\big(H_\bullet, \tilde{\partial}_H\big) \underset{\tilde{\phi} = (\mathrm{id} - \eta\delta)^{-1}\phi}{\overset{\tilde{\iota} = \iota(\mathrm{id} - \delta\eta)^{-1}}{\leftrightarrows}} (V_\bullet, \partial + \delta) \circlearrowright \tilde{\eta} = \eta(\mathrm{id} - \delta\eta)^{-1}$$

with $\tilde{\partial}_H = \partial_H + \iota(\mathrm{id} - \delta\eta)^{-1}\delta\phi$. Note: $(\mathrm{id} - \eta\delta)^{-1} = \mathrm{id} + \eta(\mathrm{id} - \delta\eta)^{-1}\delta$. **Proof of HPL:** Check some eqns.

Cor: If $H_\bullet = H_0$, then $\partial_H = \tilde{\partial}_H = 0$, so $\mathrm{H}_\bullet(V)$ does not deform. If also $V_\bullet = V_{\geq 0}$, then $\delta\phi = 0$ so $\tilde{\phi} = \phi$. These restrictions also imply $\iota, \tilde{\iota}$ are indep of choice of η.

Rmk: HPL is easy version of homotopy transfer theory: it transfers "having a Maurer–Cartan element."

Application: $H_\bullet = \mathrm{H}_\bullet(V, \partial)$, with V as above, $\partial = \partial_{\text{cl}}$, and $\delta = \partial_{\text{Leb}}$. Then $\mathrm{H}_0(V, \partial_{\text{cl}}) = \mathcal{O}(\{\mathrm{d}s = 0\})$. DGCA $(V_\bullet, \partial_{\text{cl}})$ is the *derived critical locus of s*. In general, $\mathrm{H}_\bullet(V, \partial_{\text{cl}}) = \mathcal{O}(\Pi \mathrm{T}^* \{\mathrm{d}s = 0\})$. Generic $s \Rightarrow \{\mathrm{d}s = 0\}$ is zero-dim scheme $\Rightarrow \mathrm{H}_\bullet = \mathrm{H}_0$. **Bezout's Thm** $\Rightarrow$ For generic degree-d s, $\dim \mathcal{O}(\{\mathrm{d}s = 0\}) = (d-1)^n$.

Write $s = s^{(d)} + s_{\text{sub}}$ with $s^{(d)}$ homogeneous degree-d

and $\deg s_{\text{sub}} < d$. Then $\partial_{\text{cl}} = \partial_{(d)} + \partial_{\text{sub}}$. If $s^{(d)}$ defines smooth hypersurface in $\mathbb{P}^{n-1}$ then $\{\mathrm{d}s^{(d)} = 0\}$ is origin with multiplicity $(d-1)^n$. Give $V_\bullet$ an $\mathbb{N}$-grading by $|x_i| = 1$, $|\xi_i| = d-1$. Then $\partial_{(d)}$ preserves this grading, and $V_\bullet = \bigoplus$ finite-dimensional pieces. Can choose splitting $\phi_{(d)}$ and homotopy $\eta_{(d)}$ preserving grading. Then ∂_{sub} and ∂_{Leb} lower grading, so $(\partial_{\text{sub}} + \partial_{\text{Leb}})\eta_{(d)}$ is ind-nilpotent.

Cor: Choice of $\phi_{(d)} \Rightarrow$ identification of $\mathrm{H}_\bullet(V, \partial_{\text{cl}})$ and $\mathrm{H}_\bullet(V, \partial_{\text{full}})$ with $\mathrm{H}_\bullet(V, \partial_{(d)}) = \mathcal{O}(\{\mathrm{d}s^{(d)} = 0\})$.

3. Explicit formulas from an ad hoc choice

So enough to study s homogeneous. We need $\phi = \phi_{\text{cl}}$ to choose ι and $\tilde{\iota} = \int$. And want to choose η_{cl} to write formulas for $\tilde{\iota}$. Strategy: decompose ∂_{cl} yet again.

Set $s(x) = \sum_{i_1,\dots,i_d} s_{i_1\cdots i_d} \frac{x_{i_1}\cdots x_{i_d}}{d!}$. If $a_i = s_{i\cdots i} \neq 0$, set $s_{\text{diag}} = \sum_i a_i \frac{(x_i)^d}{d!}$; $s = s_{\text{diag}} + s_{\text{mix}}$; $\partial_{\text{cl}} = \partial_{\text{diag}} + \partial_{\text{mix}}$.

Consider complex $(V_\bullet, \partial_{\text{diag}})$. It factors as a CDGA:

$$\left(\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n],\ \textstyle\sum_i a_i \frac{x_i^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}\right) = \\ = \textstyle\bigotimes_i \left(\mathbb{C}[x_i,\xi_i],\ a_i \frac{x_i^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi_i}\right)$$

Two-term complex $\left(\mathbb{C}[x,\xi], a\frac{x^{d-1}}{(d-1)!}\frac{\partial}{\partial \xi}\right)$ is easy to analyze: $\mathrm{H}_1 = 0$; H_0 has basis the images of $\{x^m : m < d-1\}$. Set $\phi(\langle x^m\rangle) = x^m$ for $m < d-1$. Choice of homotopy: $\eta(x^m) = \begin{cases} 0, & m<d-1 \\ \frac{1}{a}\xi x^{m-d+1}, & m \geq d-1 \end{cases}$.

No canonical way to tensor complexes-with-homotopy. A reasonable choice for $\eta = \eta_{\text{diag}} : V_0 \to V_1$ is

$$\eta_{\text{diag}}\left(x_1^{m_1}\cdots x_n^{m_n}\right) = \begin{cases} 0, \text{ all } m_i < d-1 \\ \dfrac{\sum_i \frac{\xi_i}{a_i}\left(\frac{\partial}{\partial x_i}\right)^{d-1}}{\sum_i \binom{m_i}{d-1}}\left(x_1^{m_1}\cdots x_n^{m_n}\right) \end{cases}$$

Can extend to $V_\bullet$ preserving $\mathbb{N}$-grading $|x| = 1$, $|\xi| = d-1$. Only $\eta_{\text{diag}} : V_0 \to V_1$ will appear in formulas.

Is $(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}})$ invertible? $\mathbb{N}$-grading $\Rightarrow$ block-diagonal with countably many finite-dim blocks $\Rightarrow$ invertibility is *very general* (weaker than *generic*) condition.

Rmk: $\mathrm{H}_\bullet(V, \partial_{\text{diag}})$ is $(d-1)^n$-dimensional in $\mathbb{N}$-gradings $0,\dots,n(d-2)$. If s is smooth, homology for $(V_\bullet, \partial_{\text{cl}})$ is no bigger. So can test if $\{x_1^{m_1}\cdots x_n^{m_n} : m_i < d-1\}$ is basis for $\mathrm{H}_\bullet(V, \partial_{\text{cl}})$ by looking at first $n(d-2)$ pieces.

Cor: For generic s (degree d, homogeneous or inhomogeneous), one choice of basis for $\mathrm{H}_\bullet(V, \partial_{\text{cl}}) = \mathcal{O}(\{\mathrm{d}s = 0\})$ consists of the images of $\{x_1^{m_1}\cdots x_n^{m_n} : m_i < d-1\}$.

Use this basis to choose $\phi : H_0 \to V_0$. If s homogeneous, homotopy for $(V, \partial_{\text{cl}})$ is $\eta_{\text{diag}}\left(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}}\right)^{-1}$ with

$$\partial_{\text{mix}}\eta_{\text{diag}}(x_1^{m_1}\cdots x_n^{m_n}) = \\ = \begin{cases} 0, \text{ all } m_i < d-1; \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_{\substack{i_1,\dots,i_{d-1},j \\ \text{not all equal}}} \frac{s_{i_1\dots i_{d-1}j}}{s_{j\cdots j}} \frac{x_{i_1}\cdots x_{i_{d-1}}}{(d-1)!} \times \\ \qquad \times \left(\dfrac{\partial}{\partial x_j}\right)^{d-1} (x_1^{m_1}\cdots x_n^{m_n}). \end{cases}$$

Thm: Suppose s homogeneous. Identify $H_0 = \mathrm{H}_0(V_\bullet, \partial_{\text{full}})$ with span of $\{x_1^{m_1}\cdots x_n^{m_n} : m_i < d-1\}$. Set $p_{\text{diag}} : V_0 \to H_0$ the map that kills anything divisible by some x_i^{d-1}. The canonical projection $p_{\text{full}} : V_0 \to H_0$ is $p_{\text{full}} =$

$$p_{\text{diag}}\left(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}}\right)^{-1} \sum_{\ell \geq 0}\left(\partial_{\text{Leb}}\eta_{\text{diag}}\left(\mathrm{id} - \partial_{\text{mix}}\eta_{\text{diag}}\right)^{-1}\right)^{\ell}$$

where $\partial_{\text{mix}}\eta_{\text{diag}}$ is as above (preserves degree) and

$$\partial_{\text{Leb}}\eta_{\text{diag}}(x_1^{m_1}\cdots x_n^{m_n}) = \\ = \begin{cases} 0, \text{ all } m_i < d-1 \\ \dfrac{1}{\sum_i \binom{m_i}{d-1}} \displaystyle\sum_i \frac{1}{a_i}\left(\frac{\partial}{\partial x_i}\right)^{d} (x_1^{m_1}\cdots x_n^{m_n}) \end{cases}$$

If s is inhomogeneous, replace $\partial_{\text{Leb}} \rightsquigarrow \partial_{\text{Leb}} + \partial_{\text{sub}}$.

4. Comments on inverse problem and applications

For fixed n, d, there are $\binom{n+d-1}{d-1}$ parameters in homogeneous s and $(d-1)^n$ contours. In principal, if you know $\sim n^{d-1} + (d-1)^n$ integrals, and can solve complicated polynomial equations, you can recover the rest.

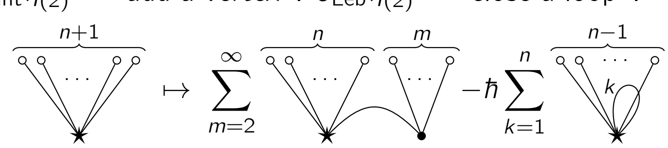
In general case, I expect this is best possible. We hope to find special cases with answers to question 3. But sending $n \to \infty$ is probably impossible in general.

Most important application: nonperturbative integrals in TQFT. Main example: Wilson loops for Chern–Simons theory (should $\Rightarrow$ HOMFLY). *This is not an infinite-dim problem!* When you include gauge symmetry, version of $(V_\bullet, \partial_{\text{cl}})$ is not quite elliptic but $\mathrm{H}_\bullet$ still manageable sized.

5. Same techniques give Feynman diagrams

Usual situation is to replace $s \rightsquigarrow s/\hbar$ for $\hbar$ formal, and ask for $\langle f\rangle$ for f supported near some nondeg critical point of s. Then use $\mathbb{C}[\![x_1,\dots,x_n]\!]$ in place of $\mathbb{C}[x]$. Quadratic piece $s^{(2)}$ dominates, and $\mathrm{H}_\bullet(V, \partial_{(2)})$ is one-dimensional. Local coordinates $\Rightarrow \eta_{(2)}$, and $(\partial_{\text{int}} + \partial_{\text{Leb}})\eta_{(2)}$ is pronilpotent: $\sum_\ell((\partial_{\text{int}} + \partial_{\text{Leb}})\eta_{(2)})^\ell$ converges.

$\partial_{\text{int}}\eta_{(2)} =$ "add a vertex". $\partial_{\text{Leb}}\eta_{(2)} =$ "close a loop".



Iterate and apply $p_{(2)} \Rightarrow$ sum of Feynman diagrams.