Classification of Topological Orders

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These slides: categorified.net/CMS2021.pdf

Acknowledgments

Many aspects of this story were first explained by L. Kong, T. Lan, X-G Wen, and their collaboration team.

I will also report on works jt with D. Gaiotto, with M. Hopkins, with D. Reutter, and with M. Yu.

Technical disclaimer

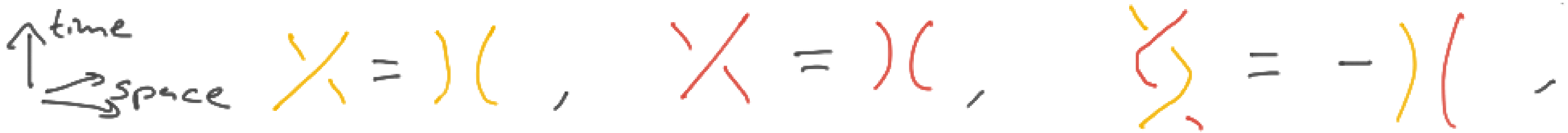
The theory of weighted colimits in presentable strict 1-categories is very well understood and robust. I assume this basic theory also applies to presentable $(\infty,1)$-categories, but this has not been carefully verified in the literature.

Motivating the definition

<u>Strategy</u>: Axiomatize and classify topological phases of matter in terms of their higher algebras of extended observables.

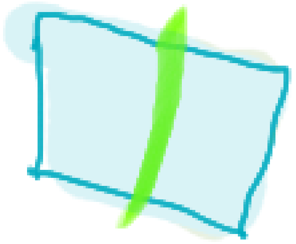
<u>Everyone's favorite example</u>: In the 2+1D toric code:

- point operators = "instantons" = $\mathbb{C}$. algebra
- line operators = "particles" = 1, e, m, $e \otimes m$, category



etc.

- surface operators = "strings" = [complicated]



Interfaces = morphisms: {strings} form a 2-category.

Fusion of strings in 2+1 D: {strings} form a monoidal 2-category

Ansätze for topological order

$\Omega := \mathrm{End}(\text{unit})$

A topological order is an operationally-defined topological phase.
An n+1 D top. order is a monoidal n-category $\mathcal{A}$ s.t.:

Ansatz 1: $\mathcal{A}$ is $\mathbb{C}$-linear, additive, and Karoubi complete. ← so you can form superpositions, composite particles, etc.

Ansatz 2: $\mathcal{A}$ is (highly) dualizable ← can fold a top. order on itself, and fold the fold, ...

Ansatz 3: $\Omega^n \mathcal{A} = \mathbb{C}$ ← robust against local perturbations.

Ansatz 4: $Z(\mathcal{A}) = n\mathrm{Vec}$ ← every operator insertion can be "remotely" detected by another operator.

nondegeneracy

$1+2 =$ "multifusion n-category"

$1+2+3 =$ "fusion n-category"

$1+2+4+(n=0) =$ "central simple algebra".

General theorems

Idea: $\exists$ "cochain complex" w/ n-cochains = fusion n-cats and $\partial = \Sigma Z$.

Suppose $\mathcal{A}$ is a (multi) fusion n-category, i.e. satisfies Ansätze 1+2:

(1) Then can (functorially) construct:

- a bulk $n+2$D phase
- a boundary $n+1$D phase

"phase" = both a functorial fully extended TQFT and a commuting projector Hamiltonian lattice realization.

and obs(boundary) = $\mathcal{A}$, obs(bulk) = $\Sigma Z(\mathcal{A})$.

Σ = "finite dim modules"

(2) $\mathcal{A}$ is nondegenerate, i.e. Ansatz 4, iff bulk is invertible:

$$\text{Topological orders} = \frac{\text{TQFTs}}{\text{invertibles}}$$

(3) Ansatze 3+4 together: $\mathcal{A} = \Sigma \Omega \mathcal{A}$, i.e. determined by the "modular tensor $(n-2)$-cat" of codim ≥ 2 operators.

Idea: no point ops $\Rightarrow$ cannot "detect" codim 1 ops $\Rightarrow$ they are all "Cheshire".

Classification Strategy

e.g. $n+1 = 3+1$

For $n+1$ D topological order $\mathcal{A}$, set $k := \lfloor \frac{n-1}{2} \rfloor$
Look at k-category $\mathcal{C} \subseteq \mathcal{A}$ of operators of dim $\leq k$, i.e.:

codimension $\geq$ dimension $+ 2$.

Enough room to unlink $\Rightarrow$ this $\mathcal{C}$ is symmetric monoidal.

Best case scenario: $\mathcal{C} \simeq k\mathrm{Rep}(\mathcal{G})$ for some k-group $\mathcal{G}$.
i.e. $\mathcal{C} \simeq$ Wilson operators for $\mathcal{G}$-gauge-theory.

If so, can "condense $\mathcal{C}$"
new top. order $\mathcal{A}/\!/\mathcal{C}$ s.t.:

- $\mathcal{G} \circlearrowright \mathcal{A}/\!/\mathcal{C}$, and $\mathcal{A}$ = gauged thy.
- only (non-cheshire) operators in $\mathcal{A}/\!/\mathcal{C}$ are in dim d s.t. $k < d < n-k$.

but there is at most one such d!

Strategic Conclusion

If can identify $\mathcal{C} \cong K\mathrm{Rep}(\mathcal{G})$, then:

n odd: $\mathcal{A}$ is canonically a higher group gauge theory.
These have a classification by generalized cohomology.

n even: $\mathcal{A}$ is canonically the result of gauging a higher group action on a TQFT whose only non-cheshire operators are in dim $\frac{n}{2}$.

$\hookrightarrow$ n even > 2: Such TQFTs are always "abelian".
These have a cohomological classification.

n = 2 case $\Leftrightarrow$ MTCs, which are completely wild and will never be classified (I expect).

Missing ingredient

some symmetric fusion k-category

How often is $\mathcal{C} \cong k\mathrm{Rep}(\mathcal{G})$ for some k-gp $\mathcal{G}$?

This is the subject of (higher) Tannaka duality.

Thm (Deligne for $k=1$): When $k \leq 2$, $\mathcal{C} = k\mathrm{Rep}(\mathcal{G})$ iff $\mathcal{C}$ is "all boson".
Otherwise, it is a twisted version of $k\mathrm{Rep}(\mathcal{G})$.

The twisting when $k \leq 2$ is related to spin structures.
In general, it is an action on $\mathcal{G}$ by spacetime symmetries.

$\Rightarrow$ except in 2+1D, topological orders should have a complete cohomological classification as "crystalline higher gauge theories".