The Universal Target Category

Theo Johnson-Freyd, Dalhousie & Perimeter

Based on jt wk in progress w/ David Reutter

These slides: categorified.net/CMSA-Colloquium.pdf

Why fermions?

Consider a quantum physical system with some species of "particles". Maybe they are quarks and leptons, or maybe they are localized excitations (Cooper pairs, phonons, ...) in some material. They can move around, and collide to form new particles.

Question: Can you assign, to each particle species X, Y, ..., a ~~Hilbert~~ vector space $\mathcal{H}(X)$, $\mathcal{H}(Y)$, ... of "internal states"?

Requirements:

- System w/ particle X and particle Y $\longmapsto \mathcal{H}(X) \otimes \mathcal{H}(Y)$.
- Superposition of X or Y $\longmapsto \mathcal{H}(X) \oplus \mathcal{H}(Y)$
- All physical processes are represented by ~~unitaries~~ isomorphisms.

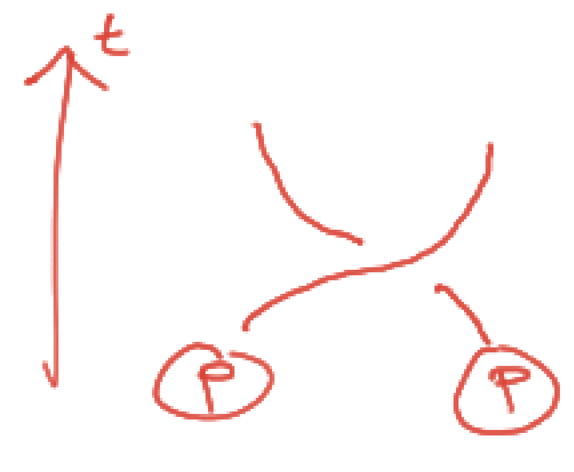
In other words: consider the category of particle configurations. Does this category admit a nice functor to ~~Hilb~~ Vec?

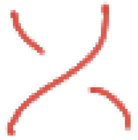
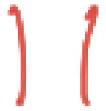
Why fermions?

Answer: No. **E.g.:** Take two protons and prepare them in the same chosen state. Total system should be in state

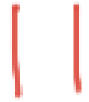
$$|\text{chosen state}\rangle \otimes |\text{chosen state}\rangle \in \mathcal{H}(\text{proton}) \otimes \mathcal{H}(\text{proton}).$$

Now run the process that switches their locations.



Should take $|\text{chosen}\rangle \otimes |\text{chosen}\rangle \mapsto |\text{chosen}\rangle \otimes |\text{chosen}\rangle$, and so if you allow both  and , the superposition of these processes should interfere constructively.

But in fact, they interfere destructively.

In other words, must have  = −  for pairs of particles in identical states.

Why fermions?

Answer: No. **Cheap fix [Dirac?]:** Super Vector spaces.

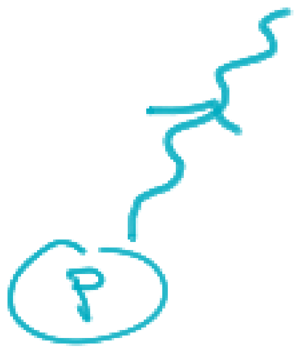
Defn: A super vector space is a $\mathbb{Z}/2$-graded vector space $V_{\bar 0} \oplus V_{\bar 1}$. Tensor product adds gradings (mod 2). Only difference: if you want to compare $v_{\bar a} \otimes w_{\bar b} \in V_{\bar a} \otimes W_{\bar b} \subseteq (V_{\bar 0} \oplus V_{\bar 1}) \otimes (W_{\bar 0} \otimes W_{\bar 1})$ to $w_{\bar b} \otimes v_{\bar a}$, you compare them with a factor of $(-1)^{\bar a \cdot \bar b}$.

In other words, you change the meaning of commutator.

Theorem [Deligne]: The cheap fix suffices*. Specifically, if $\mathcal{A}$ is any symmetric monoidal category which is not too large, and if $\mathcal{A} \neq 0$, then $\exists$ a sym mon functor $\mathcal{A} \to \mathrm{sVec}$.

* in at least 3+1 dimensions, so that .

Why $\sqrt{-1}$?



From a relativistic perspective, particles are 1-dimensional because they trace a path as they move. Every particle gives a 1D observable that measures the expected work accrued by a given path.

There are also 0D observables — instantons — that measure values of fields at moments in spacetime.

An observable is an integral of motion if its value is locally constant in space and time.

Integrals of motion form a commutative algebra*:

* in at least 1+1 dimension

Why $\sqrt{-1}$?

Question: Can you assign, to each formally-real 0D observable, a real number so that multiplication of observables $\leftrightarrow$ multiplication of numbers?

Answer: No. **E.g.:** There are superconducting materials which spontaneously generate magnetic fields. This breaks time-reversal symmetry. The ultraviolet T-symmetry instead becomes a formally-real 0D observable J s.t. $J^2 = -1$.

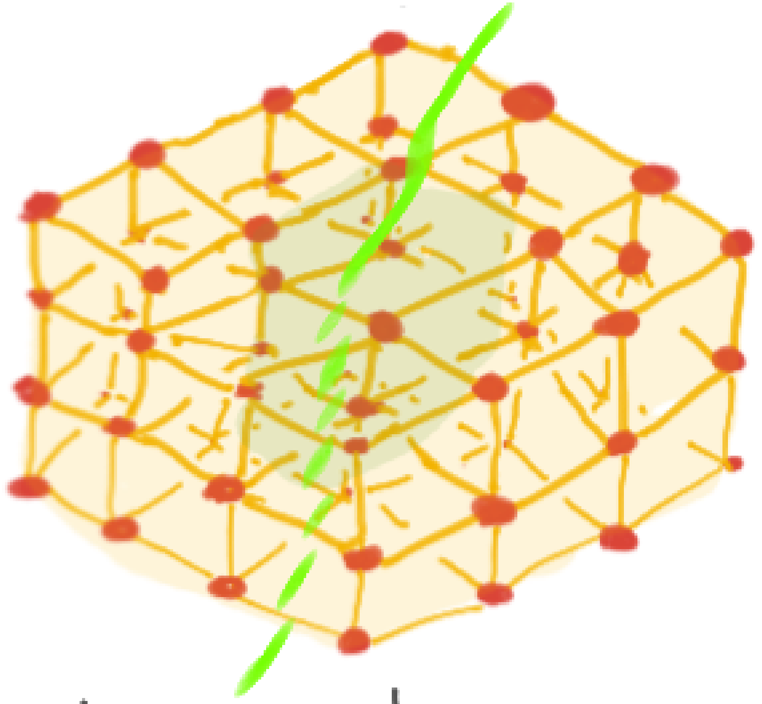
Cheap fix [Cardano?]: A ~~supernumber~~ complex number is a $\mathbb{Z}/2$-graded number $a_{\bar{0}} \oplus a_{\bar{1}}$. Multiplication adds gradings, but with an extra sign:

$$(a_{\bar{0}} \oplus a_{\bar{1}}) \cdot (b_{\bar{0}} \oplus b_{\bar{1}}) = \sum_{\bar{i},\bar{j} \in \mathbb{Z}/2} (-1)^{\bar{i}\bar{j}} a_{\bar{i}} b_{\bar{j}}.$$

Theorem [Hilbert]: The cheap fix suffices: for any commutative $\mathbb{R}$-algebra $A \neq 0$ which is not too large, $\exists$ homomorphism $A \to \mathbb{C}$.

Going higher

In addition to instantons and particles, quantum systems can have **extended objects**. E.g.: Imagine a crystalization process in 3+1D of a chemical that likes to form a cubic lattice. If it crystalizes from the outside in, it might get stuck with defects where the crystal is off by one as it goes around.



Although this costs energy, the system cannot transition into a better configuration w/o a massive, energy-expensive, change all the way to ∞: the defect is **topologically protected**. On the other hand, the system will try to straighten out bends in the defect $\rightsquigarrow$, so the defect behaves dynamically like a **vibrating string**. Even higher dimension: **(mem)branes**.

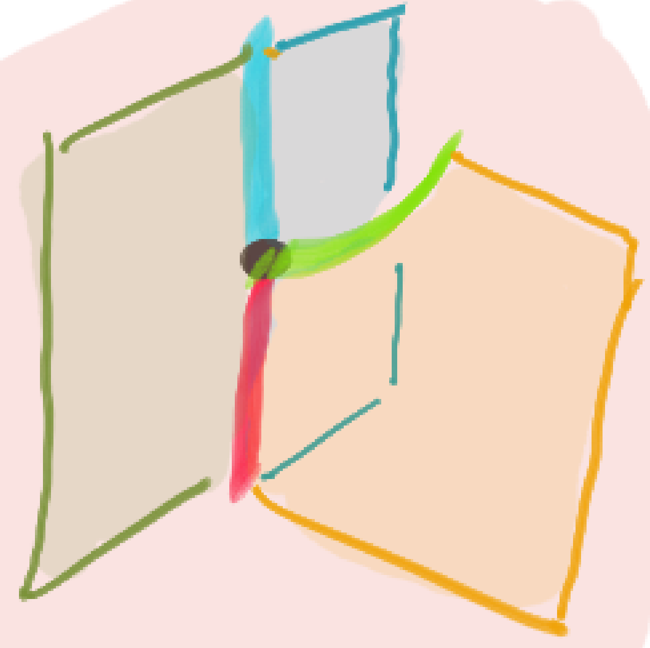
Going higher

The correct language to describe the fusion and statistics of extend objects is higher category thy.

nD objects live in an n-category:

1-morphisms are $(n-1)$D junctions between objects.
2-morphisms are junctions between junctions.
Etc.

Instanton = 0D
particle = 1D
string = 2D
...



Main Question: Is there an nD generalization of $\mathbb{C}$, sVec, ...? I.e. an n-category $\mathcal{W}^n$ s.t. every not too large non-zero commutative* n-category maps to it?

Main Theorem [JF-Reutter]: YES. And we explicitly construct it.

* aka symmetric monoidal. nD objects in at least 2n+2 D.

How to build $\mathcal{W}^\bullet$?

"not too large": this extension is reasonable.

Just like $\mathrm{sVec} \supseteq \mathrm{Vec} = \mathrm{Mod}(\mathbb{C})$, $\mathcal{W}^n \supseteq \mathrm{Mod}(\mathcal{W}^{n-1})$.

Want $\mathcal{W}^n$ nullstellensätzian aka algebraically closed.

So we are looking the algebraic closure of $\mathcal{A}^n := \mathrm{Mod}(\mathcal{W}^{n-1})$.

How to build algebraic closures? Look for Galois extensions.

How to build Galois extensions? Representation theory:

$$\left\{\begin{array}{c}\text{Galois extensions of } \mathbb{K} \\ \text{with Galois gp } G\end{array}\right\}_{/\text{iso}} \Longleftrightarrow \left\{\text{surjections } \mathrm{Gal}^{abs}(\mathbb{K}) \xrightarrow{f} G\right\}_{/\text{conj.}}$$

$$\left\{\begin{array}{c}\text{Sym mon functors} \\ \mathrm{Rep}_{\mathbb{K}} G \xrightarrow{F} \mathrm{Vec}_{\mathbb{K}}\end{array}\right\}_{/\text{iso}} \longleftrightarrow \left\{\text{maps } \mathrm{Gal}^{abs}(\mathbb{K}) \xrightarrow{f} G\right\}_{/\text{conj.}}$$

Specifically, given F, look at the commutative algebra $F(\underbrace{\mathcal{O}(G)}_{\hookrightarrow \text{ fns on } G})$.
Iff F corresponds to a surjection, then $F(\mathcal{O}(G))$ is a field, and $\mathbb{K} \hookrightarrow F(\mathcal{O}(G))$ is Galois w/ Galois gp G.

How to build $\mathcal{W}^\bullet$?

$$\left\{\begin{matrix}\text{Galois extensions of } \mathbb{K} \\ \text{with Galois gp } G\end{matrix}\right\}/\text{iso}$$
$$\updownarrow$$
$$\left\{\begin{matrix}\text{Sym mon functors} \\ \mathrm{Rep}_{\mathbb{K}} G \xrightarrow{F} \mathrm{Vec}_{\mathbb{K}}\end{matrix}\right\}/\text{iso}$$

The search for such functors is easier if $\mathrm{Rep}_{\mathbb{K}} G$ is easy. Easiest case: G is abelian, and $\mathbb{K}$ has nth roots of unity for $n = \text{exponent}(G)$.

In this case, $\mathrm{Rep}_{\mathbb{K}} G$ is generated by its invertibles $G^\vee = \hom(G, \mathbb{K}^\times)$, i.e. it is a group algebra: $\mathrm{Rep}_{\mathbb{K}}(G) = \mathrm{Vec}_{\mathbb{K}}[G^\vee]$. Thus:

$$\hom_{\text{sym}\otimes}(\mathrm{Rep}_{\mathbb{K}} G, \mathrm{Vec}_{\mathbb{K}}) = \hom_{\substack{\text{higher ab gps} \\ \text{aka spectra}}}(G^\vee, \underbrace{\mathrm{Vec}^\times_{\mathbb{K}}}_{= B\mathbb{K}^\times = \Sigma H\mathbb{K}^\times})$$

Unpacking this leads to Kummer Theory: if there are enough roots of unity, then $\{\text{abelian extensions}\} \subseteq$ Stable Homotopy Thy.

Moreover, if $G \mapsto \hom_{Sp}(HG^\vee, \Sigma H \mathbb{K}^\times)$ is (pro) representable, then you get a universal Kummer extension.

How to build $\mathcal{W}^\bullet$?

Higher Kummer Theory:

① If $\mathcal{W}^n$ is algebraically closed, then $\mathrm{Mod}(\mathcal{W}^n)$ admits a universal Kummer extension.

② This universal Kummer extension is the algebraic closure $\mathcal{W}^{n+1}$.

Why the Galois gp is abelian? Moral reason: it is π_{n+1}(something).

Higher Kronecker-Weber:

$\mathbb{Q}/\mathbb{Z} : I\mathbb{Q}/\mathbb{Z} :: \mathbb{Z} : \mathbb{S}$.

Recall: $\mu(\mathbb{C})$ = roots of unity $\cong \mathbb{Q}/\mathbb{Z}$.

① If $\mathcal{W}^\bullet$ is algebraically closed, then $\mu(\mathcal{W}^\bullet) = I\mathbb{Q}/\mathbb{Z}$.

② If $\mathcal{W}^1 = \mathrm{sVec}$ and $\mu(\mathcal{W}^\bullet) = I\mathbb{Q}/\mathbb{Z}$, then $\mathcal{W}^\bullet$ is alg. closed.

Moral reason: $\mathcal{W}^1$ knows π_1; $\mu(\mathcal{W}^\bullet)$ knows $H_\bullet$; Hurewicz.

Computing the higher Galois Group

In ordinary Galois theory, the cyclotomic character is the map

$$\mathrm{Aut}(\overline{\mathbb{K}}) \to \mathrm{Aut}(\mu(\overline{\mathbb{K}})) = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^\times.$$

The higher cyclotomic character is

$$\mathrm{Aut}(\mathcal{W}^\bullet) \to \mathrm{Aut}(\mu(\mathcal{W}^\bullet)) = \mathrm{Aut}(I\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{S}}^\times.$$

Theorem: The fibre of this map admits a description in terms of surgery and L-theory, making it look very much like the j-homomorphism $PL(\infty) \to \mathbb{S}^\times$.

Conjecture: The j-homomorphism factors through the cyclotomic character. Physically, the conjecture is a higher spin-statistics.

Computing the higher Galois Group

Sample Physical Corollary: A framed* QFT w/ nontrivial gravitational anomaly is necessarily gapless in the IR, unless the anomaly is an Arf-Kervaire invariant.

Equivalent statement: A nontrivial invertible phase of framed* matter necessarily has conducting edge modes, unless the phase is an Arf-Kervaire invariant.

Note [Hill-Hopkins-Ravenel]: The nD Arf-Kervaire invariant is framed-trivial unless $n = 2, 6, 14, 30, 62,$ and maybe 126.

* No Lorentz invariance. Equivalences can break Lorentz invariance.

Computing the higher Galois Group

Addendum:

Surgery theory fails in 3D/4D due to nonabelian knotting aka anyons.

In usual surgery, this causes the Kirby-Siebenman invariant.

In our story, there are infinitely many KS invariants, controlled by the quantum Witt gp of modular tensor categories.

The precise relation requires further study.

THANKS!

Credits: Hopkins, Schlank, Freed, Scheimbauer, Teleman, ...