Minimal nondegenerate extensions and an anomaly indicator

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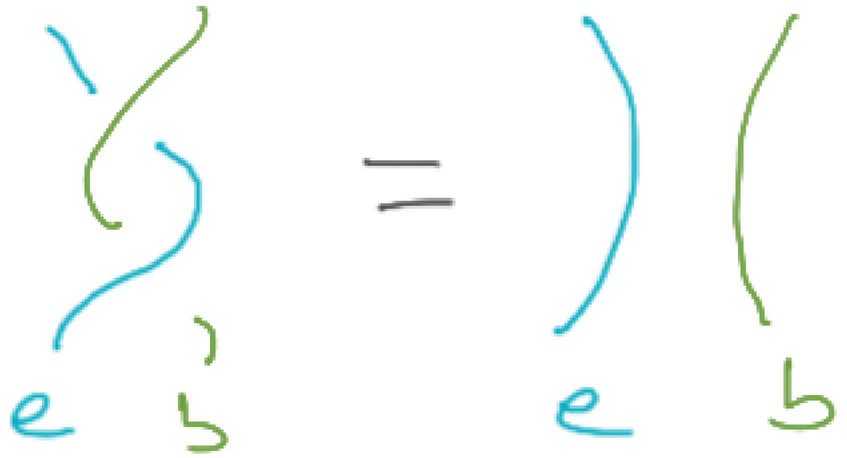
These slides: http://categorified.net/CMSA.pdf

Based on arXiv:2105.15167 jt w/ D. Reutter.

I

A braided fusion category $\mathcal{B}$ is a theory of (topological) anyons in 2+1 D: a collection of anyons and their fusion and braiding data.

It is allowed for some anyons $e \in \mathcal{B}$ to be transparent:



$\forall\, b \in \mathcal{B}$.

The subcategory $\mathcal{E} \subseteq \mathcal{B}$ of transparent anyons is called the Müger center of $\mathcal{B}$.

$\mathcal{B}$ is called nondegenerate when $\mathcal{E} = \mathrm{Vec}$.

I It is a theorem that the full collection of all anyons in any 2+1D topological phase (for any definition) is always nondegenerate (and any nondeg. BFC arises as a full thy of anyons in some topological phase).

Question: Given some BFC $\mathcal{B}$, can you represent it as some of the anyons in a 2+1 D phase?
I.e. is there a fully faithful embedding $\mathcal{B} \hookrightarrow \mathcal{M}$ with $\mathcal{M}$ nondegenerate?

Answer: Yes. Functorially construct the Drinfeld center (aka double) $Z(\mathcal{B})$ of $\mathcal{B}$. Walker-Wang give com. projector Ham. model. Braiding selects $\mathcal{B} \hookrightarrow Z(\mathcal{B})$.

I $\mathcal{B} \hookrightarrow Z(\mathcal{B})$ is unsatisfying because typically $Z(\mathcal{B})$ has <u>lots</u> of anyons transparent to $\mathcal{B}$.

Definition: A nondegenerate extension $\mathcal{B} \hookrightarrow \mathcal{M}$ is *minimal* if the only transparent-to-$\mathcal{B}$ anyons in $\mathcal{M}$ are the absolutely obvious ones $\mathcal{E} \subseteq \mathcal{B}$.

Müger 2003: Maybe every BFC has a minimal nondegenerate extension?

Drinfeld (2005?): Explicit counterexample.

Last two decades: Many papers investigating when min nondeg ext exists, and counting how many.

II To fully classify (minimal) nondegenerate extensions, it is helpful to rephrase things in 3+1D. Any BFC $\mathcal{B}$ determines a 3+1D/2+1D bulk/boundary system, with an explicit commuting projector Hamiltonian (Walker-Wang) model.

The boundary observables "are" $\mathcal{B}$. More precisely, there are the worldlines of anyons in $\mathcal{B}$, and also the worldsheets of "cheshire strings" built by condensing algebra anyons. I will call the fusion 2-category of boundary strings $\Sigma\mathcal{B}$.

The bulk anyons are the Müger centre $\mathcal{E} \subseteq \mathcal{B}$. There are also bulk string excitations. They are not all cheshire.

The braided fusion 2-category of bulk strings is the 2-categorical Drinfeld centre $\mathcal{Z}(\Sigma\mathcal{B})$.

II **Theorem:** $\dfrac{\text{bulk strings}}{\text{condensation}}$ = algebra homomorphisms from the fusion ring of $\mathcal{E}$ to the ground field.

"$Z(\Sigma\mathcal{B})$ has a nondeg. S-matrix"

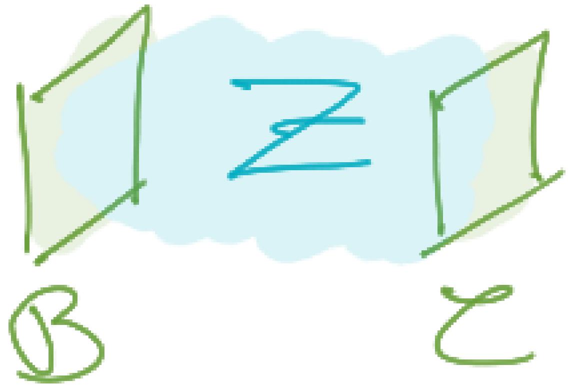
Theorem: For each equivalence $Z(\Sigma\mathcal{B}) \simeq Z(\Sigma\mathcal{C})$, can build a nondegenerate extension

$$\mathcal{B} \hookrightarrow \mathcal{M} \hookleftarrow \mathcal{C}$$

also converse

s.t. $\mathcal{C}$ = anyons in $\mathcal{M}$ transparent to $\mathcal{B}$.

Pf: Build slab



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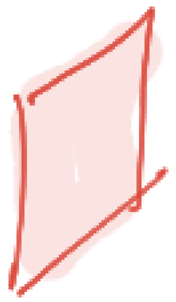


$\mathcal{M} = \mathcal{B} \otimes_{Z} \mathcal{C}$.

These theorems are straightforward to prove with even-higher categories.

With D. Reutter, we also give purely 2-categorical proofs.

III In particular, $\mathcal{B}$ has a minimal nondegenerate extension iff $Z(\Sigma\mathcal{B}) \simeq Z(\Sigma\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{B}$ is the Müger centre.

To solve the minimal nondegenerate extension problem, it suffices to classify 3+1D topological phases.

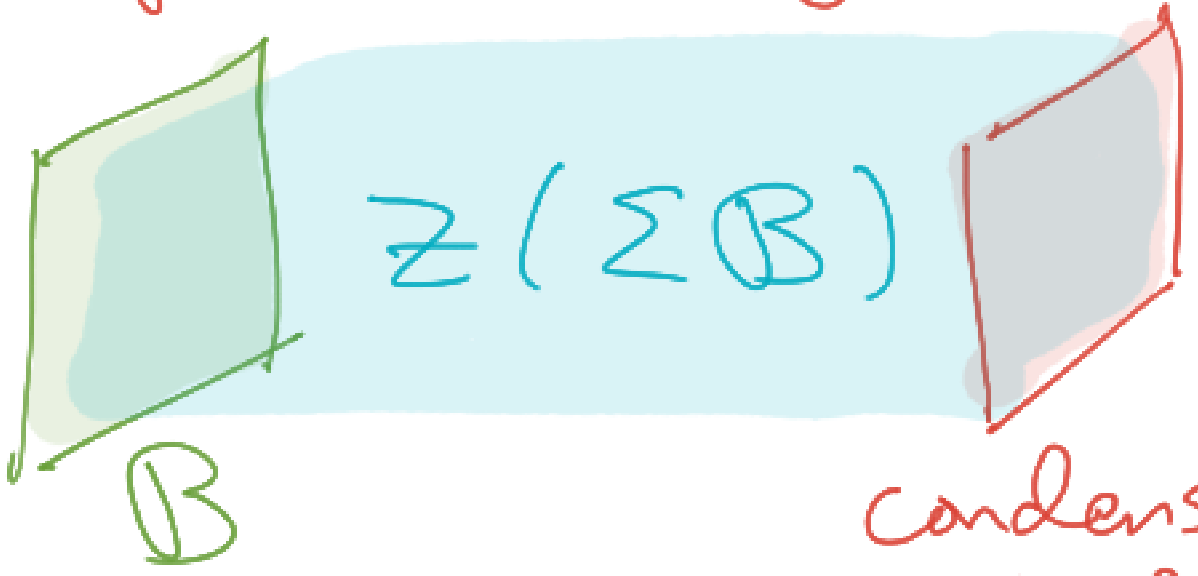
This classification was explained in beautiful work by Lan–(Kong–) Wen. The starting point is a famous theorem of Deligne's: like any symmetric fusion category $\mathcal{E}$ is either:

- $\mathrm{Rep}(G)$ for some finite gp G. "all bosons"
- an extension $\mathrm{Rep}(G) \cdot \mathrm{sVec}$. "emergent fermions"

Recall: $\mathcal{E}$ = anyons in both $Z(\Sigma\mathcal{B})$ and $Z(\Sigma\mathcal{E})$.

III If $\mathcal{E} = \mathrm{Rep}(G)$, then we can condense all the particles in $Z(\Sigma\mathcal{B})$, ending up with the vacuum 3+1D tqft.

Condensation produces a gapped interface with G-symmetry.



if $Z(\Sigma\mathcal{B}) = Z(\Sigma\mathcal{E})$, the $\Sigma\mathcal{E}$ boundary is Neumann.

condensation interface = Dirichlet boundary for G-gauge fields

To go the other way is to gauge a G-action:

$$Z(\Sigma\mathcal{B}) = G\text{-gauge thy w/ some DW action.}$$

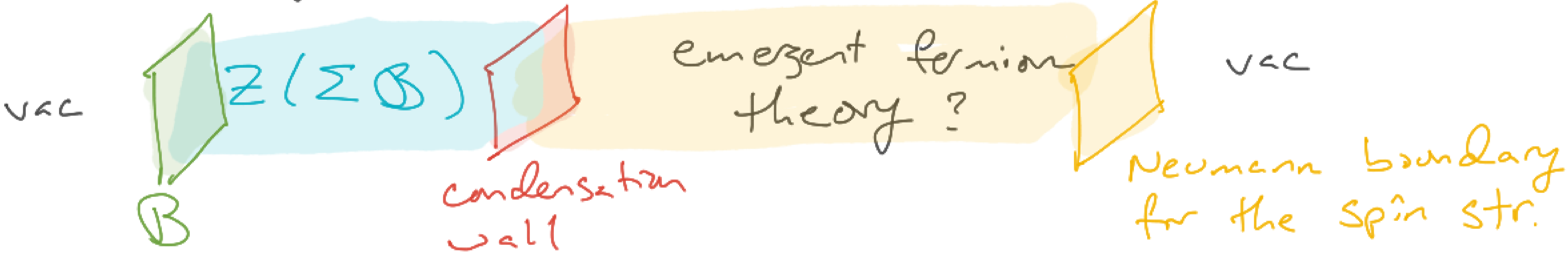
$$Z(\Sigma\mathcal{E}) = \text{vanishing DW action.}$$

DW actions $\in H^4(G; \mathbb{C}^\times)$. $\leftarrow$ obstruction to min. nondeg. ext.

Wang-Wen-Witten: all obstr's realized.

The slab $\mathcal{B} \otimes_{Z}$ condense is called the modularization of $\mathcal{B}$.

III What if $\mathcal{E} = \mathrm{Rep}(G)\cdot \mathrm{sVec}$? Can condense the bosons $\mathrm{Rep}(G) \subseteq Z(\Sigma\mathbb{B})$, End up with a 3+1D tqft whose only particle is an emergent fermion.



E.g.: might end up with the TQFT of a dynamical spin str.

Suppose we do. Then:

- $Z(\Sigma\mathbb{B})$ = Dynam. spin str. + gauge a G-action, classified by complicated but explicit generalized coh. of G.
- if "gauge action" vanishes, $\mathbb{B}$ admits min. nondeg. extension.

This reproduces Galindo-Venegas-Ramirez obstruction thy.

IV We've reduced the problem to classifying the resulting 3+1D "just a fermion" tqft. Equivalently, we've reduced to whether the slightly degenerate BFC $\mathcal{B}_G := \mathcal{B} \, Z \,$ admits a minimal nondeg. extension.

$\mathcal{B}$ condense G

i.e. Müger centre = sVec

Main theorem: It does.

this is a fun spectral sequence calculation

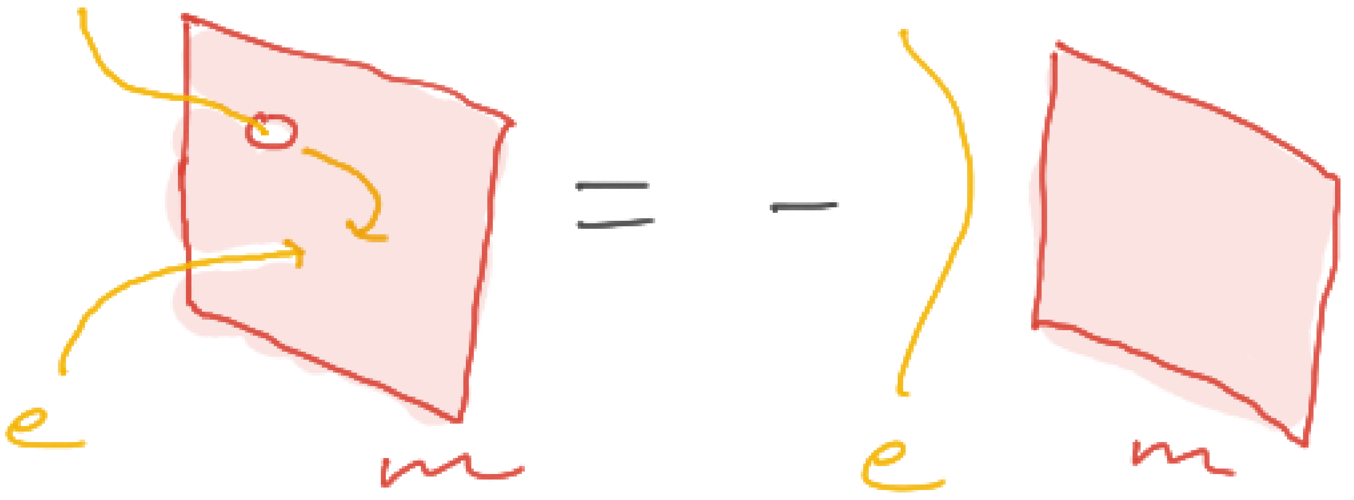
The slogan-level proof is simple.

1. There are exactly two 3+1D tqfts with only a fermion: dynamical spin str and another one.
2. The other one has a 4+1D gravitational anomaly $= (-1)^{w_2 w_3}$.
3. But $Z(\Sigma\mathcal{B})$ does not have a gravitational anomaly.

IV Converting this slogan into a rigorous proof is hard. One problem is that we want to work algebraically over an arbitrarily (algebraically closed, characteristic zero) ground field. Another is that BFCs only determine framed TQFTs, but "$(-1)^{w_2 w_3}$" is only nontrivial for oriented TQFTs. The most damning problem is that we do know how to equip the "anomalous" 3+1D TQFT with nonanomalous orientation data (and also how to assign anomalous orientation data to the "nonanomalous" one). The assignment seems to violate unitarity and spin-statistics, but so far there is not a definition of those for 2-categories of strings.

IV To give you an algebraic proof, I will use a fact which comes from the spectral sequence classification of 3+1D tqfts with only an emergent fermion "e."

In both cases, there is a (unique up to condensation) string "m" dual to e:



"magnetic" "electric"

It generates a $\mathbb{Z}_2$ 1-form symmetry.

- In the "nonanomalous" thy, this $\mathbb{Z}_2$ symmetry is nonanomalous.
- In the "anomalous" thy, this $\mathbb{Z}_2$ 1-form symmetry has a nontrivial anomaly $(-1)^{S_t^2 S_t^1} \in H^5(K(\mathbb{Z}_2, 2); \mathbb{C}^\times)$.

V So we win if we can find an anomaly indicator for $\mathbb{Z}_2$ 1-form symmetries in 3+1 D and calculate it in $Z(\Sigma\mathcal{B})$ (with $\mathcal{B}$ slightly degenerate).

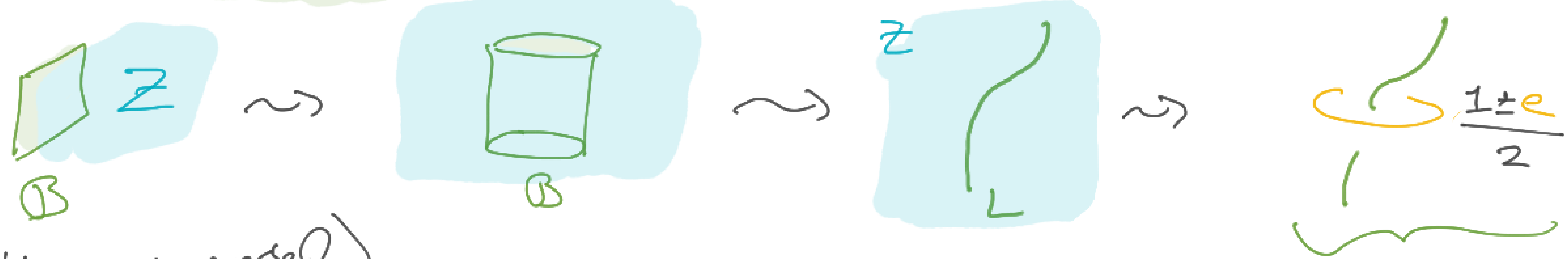
Out of thin air, the indicator:

Wrap your symmetry defect on a Klein bottle in 3+1 D with non-bounding Pin_+ str.

This indicator makes sense for any string state with a choice of iso to its dual $x \simeq x^\vee$.

orientation reversing defect, since $m \simeq m^\vee$.

VI Now for a trick. It is hard to identify $m \in Z(\Sigma \mathcal{B})$ for any specific $\mathcal{B}$. But it is easy to build a non-simple magnetic object which is canonical:



(time surpressed)

image of this idempotent is $L_\pm$.

L_- is magnetically charged, i.e. a $\oplus$ of m's.

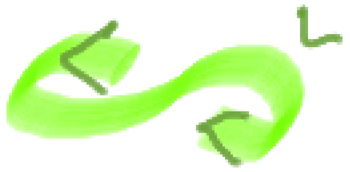
L_+ is magnetically neutral.

VI Since $L_- = \bigoplus m$'s, any orientation reversal iso $L_- \simeq L_-^\vee$ will at worst permute them. Thus:

If anomalous $\Rightarrow \mathrm{Klein}(m) = -1 \Rightarrow \overset{= -\mathrm{tr}(\text{perm matrix})}{\mathrm{Klein}(L_-)} \leq 0$.

However, a long string-diagrammatic calculation shows:

$$\mathrm{Klein}(L_\pm) = \underbrace{\tfrac{1}{2}\left(\#\, b \in \mathcal{B} \text{ s.t. } b \cong b^*\right)}_{\geq 1} \pm \underbrace{\tfrac{1}{2}\left(\#\, b \in \mathcal{B} \text{ s.t. } b \cong e \otimes b^*\right)}_{= 0}.$$

Uses: anyon  $^L = \bigoplus_{b \in \mathcal{B}} 1$, plus explicit nice choice of $L \simeq L^\vee$ over $b \mapsto b^*$.