Some thoughts on the Kitaev-Kapustin cobordism conjecture

Workshop on QFT and Top'l Phases via Homotopy Thy Operator Algs

Harvard CMSA / MPIM Bonn

Theo Johnson-Freyd, 7 July 25

Based on jt work in progress w/ David Reutter

Outline of talk

1. $\mathrm{InvPhase}^\bullet$ is a spectrum. Which one?

   Kitaev, Kapustin 2014:

   Conjecture: (bosonic case) $I_{\mathbb{Z}} MSO$?

   Freed + Hopkins 2016:

   True for reflection-positive TQFTs.

2. Universal target category $\rightsquigarrow$

   $G_n(\mathrm{Vec}_{\mathbb{C}}^\bullet) = I_{\mathbb{C}^\times} M\mathcal{Gal}$ for $\mathcal{Gal} \approx SPL$.

3. Bord, PL, and higher daggers.

4. Revised conjecture:

   $\mathrm{InvPhase}^\bullet = I_{\mathbb{Z}} MSPL$.

1. Motivating Q: Who is InvPhase?

$\mathrm{InvPhase}^n :=$ top'l space of invertible phases of gapped nD matter.

(bosonic, no T sym.)

Kitaev 2013: $\mathrm{InvPhase}^\bullet$ is an Ω-spectrum.

In other words, there is a homotopy equivalence

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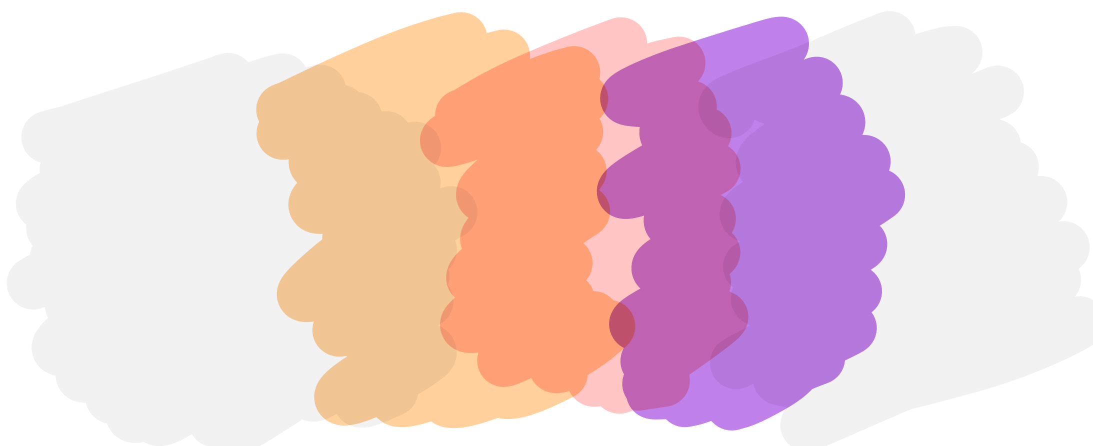
Easy Direction: ($\leftarrow$)

Given a path $\text{triv} \xrightarrow{\gamma} \text{triv}$ in InvPhase^n, run it adiabatically in space



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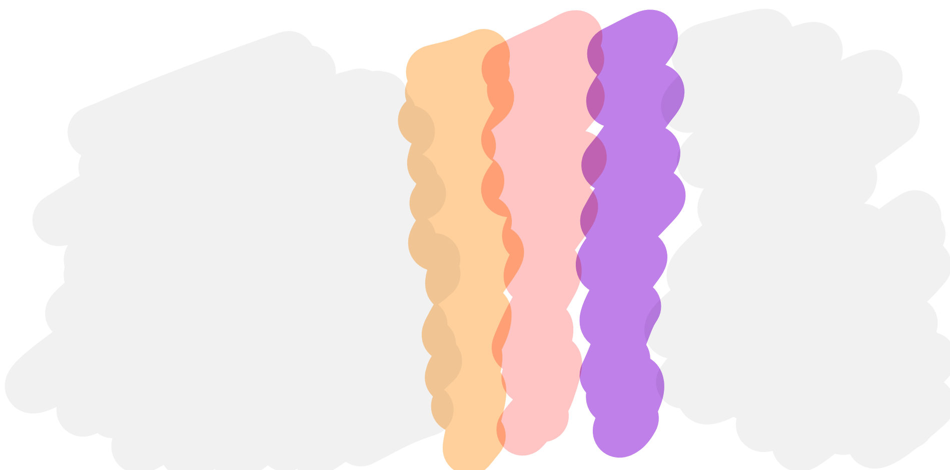

and then zoom out.

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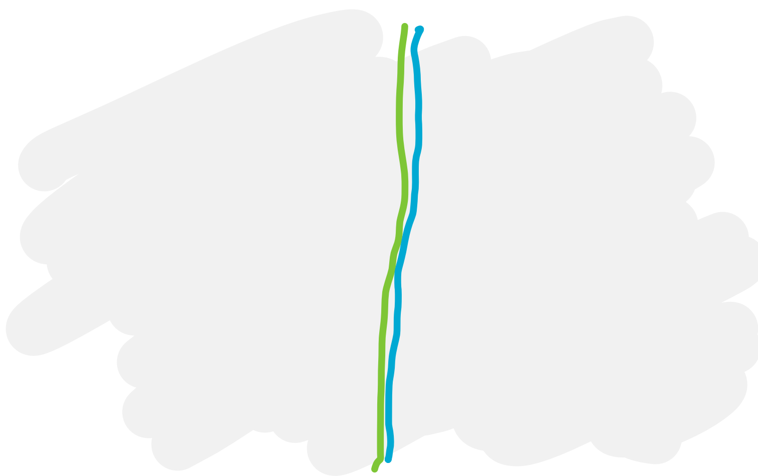

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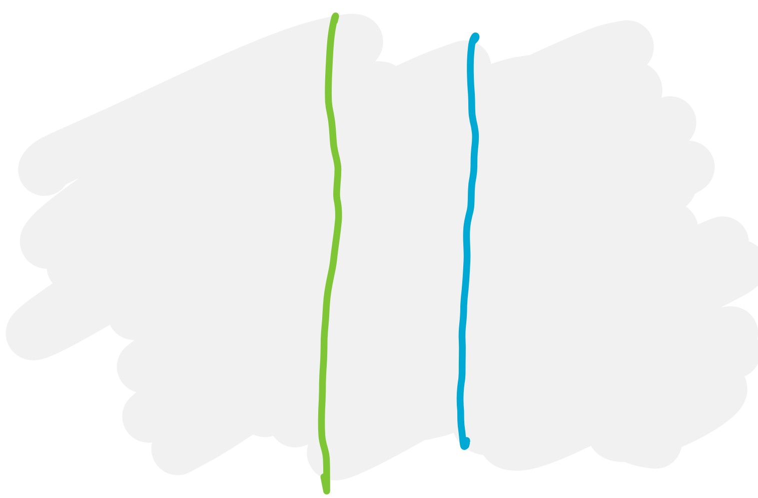

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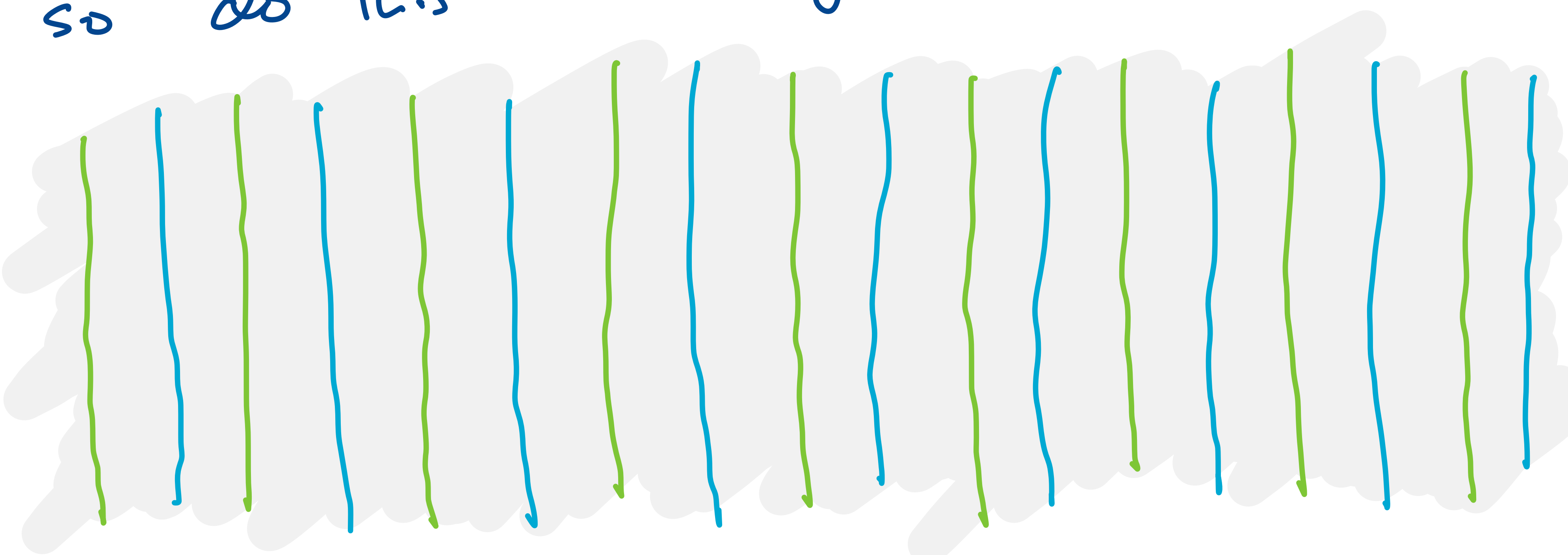

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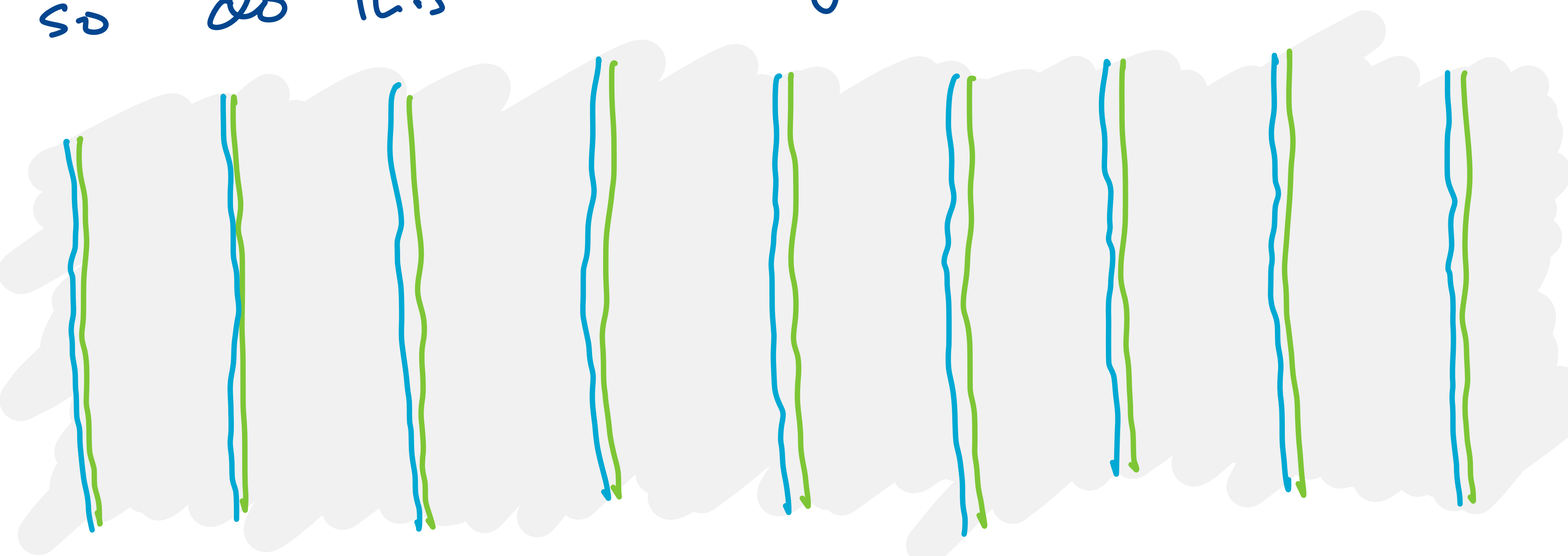

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WHICH ONE?

Conjecture (Kapustin 2014, Kitaev?)

$$\mathrm{InvPhase}^\bullet \simeq \Sigma I_{\mathbb{Z}} MSO$$

with fermions: $\Sigma I_{\mathbb{Z}} MSpin$.

with T sym: $\Sigma I_{\mathbb{Z}} MO$.

with G sym: $\Sigma I_{\mathbb{Z}} M(SO \times G)$

Experimental evidence:

Matches by hand classifications in low dim.

Kapustin 2014 Conjecture:

$$\mathrm{InvPhase}^{\bullet} \simeq \Sigma I_{\mathbb{Z}} MSO$$

Theoretical evidence (Freed–Hopkins 2016):

Ansatz: $SO(n)$ symmetry emerges in n-dimensional gapped phases.

JUSTIFY?

Ansatz: Inv phases of QFTs are determined by their partition functions.

This suggests

$$\mathrm{InvPhase}^{n} \simeq \Omega^{\infty} \Sigma I_{\mathbb{Z}} MTSO(n).$$

But $\Omega^{\infty} \Sigma I_{\mathbb{Z}} MTSO(\bullet)$ is not a spectrum.

It fails the "hard direction".

Kapustin 2014 Conjecture:

$$\mathrm{InvPhase}^{\bullet} \simeq \Sigma I_{\mathbb{Z}}^{MSO}$$

Theoretical evidence (Freed - Hopkins 2016):

Ansatz: The emergent TQFT Z of a gapped phase is reflection positive aka top-dagger.

(a) Z intertwines reflection w/ $(-)^{\dagger} := \overline{(-)}^{\vee}$.

(b) This C_2 equivariance is Hilbert on

$$\mathrm{Bord}_{n-1}^{or} \longrightarrow \mathrm{Bord}_{n}^{or}.$$

why just C_2?

Freed - Hopkins Theorem:

$$\pi_0 \left\{ \begin{array}{c} \text{dagger functors} \\ \mathrm{Bord}_n^{or} \to \Sigma I_{\mathbb{Z}} \end{array} \right\} = \pi_{-n} \Sigma I_{\mathbb{Z}}^{MSO}.$$

2. What deeper Galois theory tells us.

$B^n\mathbb{C} :=$ sym $\otimes$ n-cat w/
ob $=$ 1-mor $= \dots = (n-1)$-mor $= *$
n-mor $= \mathbb{C}$.

inc. $\oplus$'s.

$\mathrm{Vec}^n_{\mathbb{C}} :=$ higher Karoubi completion of $B^n\mathbb{C}$.

$G_m(\mathrm{Vec}^n_{\mathbb{C}}) :=$ invertible objects in $\mathrm{Vec}^n_{\mathbb{C}}$.

This is the part of $\mathrm{InvPhase}^n$ that very finite constructions can see.

e.g. gauging finite higher symmetries...
always commuting projector.

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Idea: $\mathrm{Vec}^n_{\mathbb{C}}$ is a field of depth n.

JF-Reutter (2026?): It makes sense to talk about algebraic closures of deeper fields.

$\mathcal{W}^n :=$ algebraic closure of $\mathrm{Vec}^n_{\mathbb{C}}$.

Thm. $\mathrm{Vec}^n_{\mathbb{C}} \to \mathcal{W}^n$ is Galois.

Cor: $\mathbb{G}_m(\mathrm{Vec}^n_{\mathbb{C}}) = \mathrm{Aut}_{\mathbb{C}}(\mathcal{W}^n)$-fixed points in $\mathbb{G}_m(\mathcal{W}^n)$.

Cor: $\mathbb{G}_m(\mathrm{Vec}^\bullet_{\mathbb{C}}) = \mathrm{Aut}_{\mathbb{C}}(\mathcal{W}^\bullet)$ - fixed points in $\mathbb{G}_m(\mathcal{W}^\bullet)$ $=: \mathcal{G}al$

$\simeq I^\bullet_{\mathbb{C}^\times} \times H^{\bullet - 4}_{\mathbb{Q}^\infty}$

$\mathrm{Aut}_{\mathbb{C}^\times}(I_{\mathbb{C}^\times}) = SL_1(\mathbb{S})$ ← sphere spectrum

So there is a map $\mathcal{G}al \to SL_1(\mathbb{S}) \to GL_1(\mathbb{S})$

and

$$\mathbb{G}_m \mathrm{Vec}^\bullet_{\mathbb{C}} = I_{\mathbb{C}^\times}{}^{M\mathcal{G}al}.$$

Maybe $\mathcal{G}al = SO$?

Set $\mathcal{L} :=$ L-thy of π-finite ab. gauge theories.

Theorem ("Surgery for framed TQFTs")

There is an almost exact triangle

$$\mathcal{G}al \longrightarrow SL, \mathbb{S} \longrightarrow \Sigma^\infty I_{\mathbb{C}^\times} \mathcal{L}$$

with $\underbrace{\text{cohomology}}_{= \mathrm{fib}(\mathrm{cofib}(\mathcal{G} \to S) \to I_{\mathbb{C}^\times} \mathcal{L})}$ a $K(\pi, 4)$.

Compare ("Surgery for framed manifolds")

There is an almost exact triangle

$$SPL \longrightarrow SL, \mathbb{S} \longrightarrow \text{L-thy of finite Spectra}$$

with cohomology a $K(\pi, 4)$.

3. Where did SPL come from?

* w/ (piecewise) smooth inverse

$PL(n) := \{$piecewise-smooth* homeomorphisms of $\mathbb{R}^n\}$.

$O(n) \simeq \{$smooth* homeomorphisms of $\mathbb{R}^n\}$

$PL := \operatorname{colim}_{n\to\infty} PL(n)$. $SPL :=$ preserve orientation.

Lurie: Smoothing theory $\Rightarrow$ $\mathrm{Bord}_n^{fr,sm} \xrightarrow{\sim} \mathrm{Bord}_n^{fr,PL}$.

Cobordism Hypothesis $\Rightarrow$ $\mathrm{Aut}(\mathrm{Bord}_n^{fr}) \xleftarrow{\sim} PL(n)$

if $n \neq 4$. $n=4$ case open.

From this vantage point, we could ask: why did we ever think TQFTs should be smooth?

$O(1) = PL(1) = \mathrm{Aut}(\infty$-category of $(\infty,1)$-cats$)$.

Action trivializes on subcat of $(\infty,0)$-cats.
This is what allows to talk about
dagger categories aka reflection positivity.

<u>Conjecture</u> (FHJFKMNLPRSSV 2024)

$PL(n) = \mathrm{Aut}(\infty$-cat of (∞,n)-cats w/ adjoints$)$

The conjecture allows for a deeper version of dagger categories.

<u>Conjecture:</u> $\mathcal{W}^{\bullet}$ is deeper dagger.

4. Closing thoughts

$\mathrm{Vec}^\bullet_{\mathbb{C}}$ and $\mathcal{W}^\bullet$ are too finite: they cannot capture all gapped phases, only (some?) commuting projector phases.

Kitaev's argument more generally says:

{gapped phases} is a univalent higher category.

Conjecture: It is deeper dagger.

Conjecture: There is an "analytic $\mathcal{W}^\bullet$" s.t. $\mathrm{SPL} \xrightarrow{\text{dagger action}} \mathcal{G}al$ is iso

Would imply: $\mathrm{InvPhase}^\bullet = I_{\mathbb{Z}}^{\mathrm{MSPL}}$.