Deeper-categorical algebraic closure

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https://categorified.net/Colloquium-Alberta.pdf


Why fermions?

Consider a quantum system with various species of (quasi)particles: quarks and leptons, phonons and Cooper pairs, ...

Question: Can you consistently model each species by a vector space of "internal states"? Requirements:

- Superposition of particles $\leftrightsquigarrow$ $\oplus$ of vector space.
- Fusion of particles $\leftrightsquigarrow$ $\otimes$ of vector spaces.
- Exchanging particles $\leftrightsquigarrow$ natural iso $\times : V \otimes W \cong W \otimes V$.

In other words: Does there necessarily exist a symmetric monoidal functor from the category of particle configurations to $\mathbf{Vec}$?

Warning: In ($\leq$2+1)D, particles can have nonsymmetric braiding statistics. Particle braiding is symmetric in ($\geq$3+1)D.

Why fermions?

Question: Does there necessarily exist a symmetric monoidal functor from the category of particle configurations to $\mathbf{Vec}$?

Answer: No. For any vector space, the exchange isomorphism $\times : V \otimes V \cong V \otimes V$ has positive trace: it interferes constructively with the identity isomorphism. On the other hand, for physical protons or electrons, exchanging two identical particles interferes destructively with leaving them in place.

Cheap fix (Grassmann, Dirac, Fermi): Formally declare a new type of vector — a supervector — which comes in two parts. The even part satisfies the usual exchange statistics, but the odd part picks up an extra -1 under self-exchanges.

In other words, you use the category $\mathbf{sVec}$, which is almost the same as the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, except you add a -1 to the commutator.

Why fermions?

Cheap fix (Grassmann, Dirac, Fermi): Formally declare a new type of vector — a supervector — which comes in two parts. The even part satisfies the usual exchange statistics, but the odd part picks up an extra -1 under exchanges.

Theorem (Doplicher–Roberts, Deligne): This cheap fix suffices! If $\mathcal{A}$ is any symmetric monoidal $\mathbb{C}$-linear category which is nonzero and not too large, then there exists a symmetric monoidal $\mathbb{C}$-linear functor $\mathcal{A} \to \mathbf{sVec}_{\mathbb{C}}$.

Details: For both Doplicher–Roberts and Deligne, not too large includes rigid: every object in $\mathcal{A}$ should have a dual object (e.g. dual of a finite-dimensional vector space). For Doplicher–Roberts, not too large = rigid + C*. For Deligne, not too large = rigid + finite-dimensional hom spaces + abelian + finite-length Jordan-Holder decompositions + $\mathrm{length}(X^{\otimes n}) \leq C^n$.

Why $\sqrt{-1}$?

Consider a quantum system with various local integrals of motion: local operators whose value is locally constant as a function of their location in spacetime.

Question: Can you consistently model each integral of motion by a numerical "value"? Requirements:

- Superposition of operators $\leftrightsquigarrow$ $+$ of numbers.
- Fusion of operators $\leftrightsquigarrow$ $\times$ of numbers.

In other words: Does there necessarily exist a homomorphism from the commutative ring of integrals of motion to $\mathbb{R}$?

Warning: In ($\leq$0+1)D, integrals of motion can have noncommutative operator products. Operator product is commutative in ($\geq$1+1)D.

Why $\sqrt{-1}$?

Question: Does there necessarily exist a homomorphism from the commutative ring of integrals of motion to $\mathbb{R}$?

Answer: No. The square of any real number is positive. But systems with spontaneously broken time-reversal symmetry have (real) integrals of motion that square to -1.

Cheap fix (Cardano): Formally declare a new type of number — a complex number — which comes in two parts. The real part satisfies the usual multiplication law, but the imaginary part picks up an extra -1 under self-multiplications.

In other words, you use the ring $\mathbb{C}$, which is almost the same as the ring of $\mathbb{Z}/2\mathbb{Z}$-graded real numbers, except you add a -1 to the multiplication law.

Why $\sqrt{-1}$?

Cheap fix (Cardano): Formally declare a new type of number — a complex number — which comes in two parts. The real part satisfies the usual multiplication law, but the imaginary part picks up an extra -1 under self-multiplications.

Theorem (Gauss, Hilbert, Gelfand): This cheap fix suffices! If A is any commutative $\mathbb{R}$-algebra which is nonzero and not too large, then there exists a commutative $\mathbb{R}$-algebra homomorphism $A \to \mathbb{C}$.

Details: For Gauss, not too large = singly-generated. For Hilbert, not too large = finitely-generated. For Gelfand, not too large = C*.

Punchline: $\mathbf{Vec}_{\mathbb{C}} \rightsquigarrow \mathbf{sVec}_{\mathbb{C}}$ is an algebraic closure!

$$\mathbb{R} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbb{C} \overset{\text{modules}}{\rightsquigarrow} \mathbf{Vec}_{\mathbb{C}} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbf{sVec}_{\mathbb{C}} \rightsquigarrow \dots$$

Going higher

$$\mathbb{R} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbb{C} \overset{\text{modules}}{\rightsquigarrow} \mathbf{Vec}_{\mathbb{C}} \overset{\text{alg. clos.}}{\rightsquigarrow} \mathbf{sVec}_{\mathbb{C}} \rightsquigarrow \ldots$$

n-categories organize n-(spacetime-)dimensional objects in quantum systems: strings, defects, interfaces, branes, ...

Question: What kind of statistics will we need to introduce to realize all possible n-dimensional objects? What is the algebraically closed n-category? Does it even exist?

Strategy: Set $\mathcal{W}^0 := \mathbb{C}$, $\mathcal{W}^1 := \mathbf{sVec}_{\mathbb{C}}$, $\mathcal{W}^n :=$ algebraic closure of $\mathbf{Mod}(\mathcal{W}^{n-1})$. Build these inductively.

Higher roots of unity

Lemma: If $\mathbb{K}$ is an algebraically closed field of characteristic zero, then its roots of unity $\mu(\mathbb{K}) :=$ torsion subgroup of $\mathbb{K}^\times$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$. If $\mathbb{K}$ has characteristic p, then $\mu(\mathbb{K}) \cong \mathbb{Q}_p/\mathbb{Z}_p$.
Proof: Pick a finite group A (of order coprime to p). Look at the group algebra $\mathbb{K}[A]$. There must be $|A|$-many maps $\mathbb{K}[A] \to \mathbb{K}$. But $\hom(\mathbb{K}[A], \mathbb{K}) = \hom(A, \mu(\mathbb{K}))$. This forces $\mu(\mathbb{K}) \cong \mathbb{Q}/\mathbb{Z}$.

Basically the same thing works in higher categories, with some vocabulary change:

abelian group $\rightsquigarrow$ spectrum

finite abelian group $\rightsquigarrow$ π-finite spectrum

A spectrum is basically a coherently homotopy-commutative topological group. It is π-finite when the product of its homotopy groups is finite.

Higher roots of unity

The spectrum analogue of $\mathbb{Q}/\mathbb{Z}$ is called the Brown–Comenetz dual to spheres denoted $I_{\mathbb{Q}/\mathbb{Z}}$. Its homotopy groups:

$$\pi_n I_{\mathbb{Q}/\mathbb{Z}} = \begin{cases} 0, & n > 0 \\ \hom(\pi^s_{|n|}, \mathbb{Q}/\mathbb{Z}), & n \leq 0 \end{cases}$$

$\pi^s_m := \varinjlim_{k\to\infty} \pi_{m+k} S^k =$ stable homotopy groups of spheres.

Theorem (Brown–Comenetz): Consider the homotopy 1-category of the ∞-category of those spectra A equipped with a map $\pi_0 A \to \mathbb{Q}/\mathbb{Z}$. This 1-category has a terminal object, $I_{\mathbb{Q}/\mathbb{Z}}$.

Existence requires $\mathbb{Q}/\mathbb{Z}$ is a divisible group.

Higher roots of unity

Lemma: If $\mathcal{W}^n$ is an algebraically closed symmetric monoidal n-category of characteristic zero, then its roots of unity $\mu(\mathcal{W}^n)$, i.e. the torsion subspectrum of $(\mathcal{W}^n)^\times$, is iso to $I^n_{\mathbb{Q}/\mathbb{Z}} := \Omega^\infty \Sigma^n I_{\mathbb{Q}/\mathbb{Z}}$.

Main Theorem (JF–Reutter): There is a unique sequence $\mathcal{W}^\bullet$ with:

1. $\mathcal{W}^n$ is a rigid sym mon $\mathbb{C}$-linear n-category with duals.
2. $\mathcal{W}^n$ is an ind. limit of finite separable extensions of $\mathbf{Mod}^n(\mathbb{C})$.
3. $\mathcal{W}^{n-1} \cong \mathrm{End}_{\mathcal{W}^n}(1)$, i.e. $\mathcal{W}^\bullet$ is a categorical spectrum.
4. $\mu(\mathcal{W}^n) \cong I^n_{\mathbb{Q}/\mathbb{Z}}$.

Item 2 can be weakened. Depending on the weakening, there are also transcendental extensions. Ask me in private.

Higher Kummer extensions

Suppose a field $\mathbb{K}$ has $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$. It probably is not algebraically closed. For example, it can still have noncyclotomic abelian extensions.

Theorem (Kummer): Suppose $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$ and A is a finite abelian group with $A^\vee := \hom(A, \mathbb{Q}/\mathbb{Z})$. Then A-Galois extensions of $\mathbb{K}$ are classified by $\mathrm{Ext}^1(A^\vee, \mathbb{C}^\times) = \hom(A^\vee, (\mathbf{Vec}_{\mathbb{K}})^\times)$.

Proof: An A-Galois extension $\mathbb{K} \to \mathbb{L}$ is in particular an A-representation. If $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$, then every A-representation is diagonalizable into one-dimensional eigenspaces. This diagonalization writes $\mathbb{L} = \bigoplus_{\alpha \in A^\vee} \mathbb{L}_\alpha$ as an $A^\vee$-graded commutative algebra. The assignment $\alpha \mapsto \mathbb{L}_\alpha$ is a symmetric monoidal functor $A^\vee \to \mathbf{Vec}_{\mathbb{K}}$, or equivalently a spectrum map $A^\vee \to (\mathbf{Vec}_{\mathbb{K}})^\times$.

Higher Kummer extensions

Theorem (Kummer): Suppose $\mu(\mathbb{K}) = \mathbb{Q}/\mathbb{Z}$ and A is a finite abelian group with $A^\vee := \hom(A, \mathbb{Q}/\mathbb{Z})$. Then A-Galois extensions of $\mathbb{K}$ are classified by $\hom_{\mathbf{hoSp}}(A^\vee, (\mathbf{Vec}_{\mathbb{K}})^\times)$.

Since $A^\vee$ is finite, any map to $(\mathbf{Vec}_{\mathbb{K}})^\times$ factors through its torsion subgroup $\mu(\mathbf{Vec}_{\mathbb{K}})$.

Punchline: Abelian extensions of $\mathbb{K}$ are controlled by the roots of unity in $\mathbf{Vec}_{\mathbb{K}}$.

A field without abelian extensions is solvably closed. To form the solvable closure, you keep adjoining abelian extensions until $\mu(\mathbf{Vec}_{\mathbb{K}})$ is as simple as possible.

Nonabelian simple groups are much harder. Fortunately they won't arise for us.

Using these ideas, and some long exact sequences, we showed:

Theorem (JF–Reutter): Suppose $\mathcal{W}^{n-1}$ is algebraically closed, in particular $\mu(\mathcal{W}^{n-1}) = I^{n-1}_{\mathbb{Q}/\mathbb{Z}}$. Then $\mathbf{Mod}(\mathcal{W}^{n-1})$ has a universal abelian extension: there is a universal abelian group U^n such that

$$\pi_0\{A\text{-graded extensions of } \mathbf{Mod}(\mathcal{W}^{n-1})\} = \hom_{\mathbf{AbGp}}(A, U^n).$$

Moreover, this universal U^n-extension (classified by $\mathrm{id} : U^n \to U^n$) is algebraically closed. It is $\mathcal{W}^n$.

The dual group $(U^n)^\vee = \hom(U^n, \mathbb{Q}/\mathbb{Z})$, or rather some cohomological shift of it, is the relative Galois group $\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1})$. The full Galois group $\mathbf{Gal}(\mathcal{W}^n/\mathbb{C})$ is a homotopical group with $\pi_n\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1}) = (U^n)^\vee$.

Why don't we see nonabelian extensions?

Our proof is direct and algebraic. Indeed, a priori there is no well-defined algebraic closure. We need to construct it without knowing its existence. But what seems to be going on, with hindsight, is:

- If everything works well, then definitely $\mathbf{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1}) = K(\pi_n \mathbf{Gal}(\mathcal{W}^n/\mathbb{C}), n)$ is abelian.
- Any homotopical group with trivial π_0 is solvable.
- Given a categorical spectrum $\mathcal{A}^\bullet$, its roots of unity $\mu(\mathcal{A}^\bullet)$ are some sort of "étale cohomology" of "$\mathrm{Spec}(\mathcal{A}^\bullet)$." Saying $\mu(\mathcal{W}^\bullet) = I_{\mathbb{Q}/\mathbb{Z}}$ is saying that $\mathrm{Spec}(\mathcal{W}^\bullet)$ is a homology point.
- On the other hand, the ordinary Galois group $\mathrm{Gal}(\overline{\mathcal{A}^0}/\mathcal{A}^0)$ measures "étale π_1." Saying $\mathcal{W}^0 = \mathbb{C}$ is saying that $\mathrm{Spec}(\mathcal{W}^\bullet)$ has the fundamental group of a point.
- Whitehead, Hurewicz: A space with both the homology and fundamental group of a point is a point.

Calculating the Galois group

An ordinary absolute Galois group has a cyclotomic character $j : \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^\times$. Our cyclotomic character lands in $\mathrm{Aut}(I_{\mathbb{Q}/\mathbb{Z}}) = \widehat{\mathbb{S}}^\times$.

Using ideas from TQFT, from abelian Chern–Simons theory, and surgery theory, we can show:

Theorem (JF–Reutter): The fibre of the cyclotomic character admits an invariant valued in the L-theory of π-finite spectra. This invariant an isomorphism in degrees ≥ 5.

Calculating that L-theory, we find a LES ($n \geq 5$):

$$\cdots \to \pi_{-n} I_{\mathbb{Q}/\mathbb{Z}} \to U^n \to \left\{ \begin{array}{ll} 0, & n \text{ even} \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \pmod 4 \\ \text{Witt group}, & n \equiv 3 \pmod 4 \end{array} \right\} \to \dots$$

Credit where it's due

Idea that $\mathbf{Vec}_{\mathbb{C}} \to \mathbf{sVec}_{\mathbb{C}}$ is an algebraic closure: inspired by conversations and collaborations with Chirvasitu.

Idea to look for a categorical spectrum with $\mu(\mathcal{W}) = I_{\mathbb{Q}/\mathbb{Z}}$: Hopkins, Freed–Hopkins.

$\mathbf{Mod}(\mathbf{sVec})$ is algebraically closed. Conjectured by Hopkins. Immediate consequence of my "grouplike" theorem with Yu.

Constructing $\mathcal{W}^3$: Freed–Scheimbauer–Teleman. They told us their answer, and we back-solved to find the right question.

Surgery analysis of Galois group: inspired by Lan–Kong–Wen.

Narrative in terms of Kummer extensions: inspired by Burklund–Schlank–Yuan.

THANK YOU

25 YEARS

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