Phases of SQFTs

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These slides are available at categorified.net/Dal-SQFTs.pdf

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Plan for the talk:

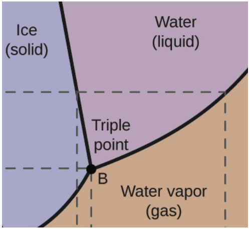
Spaces of physical systems

The Witten index

Beyond the Witten index

Mathieu Moonshine

Spaces of physical systems



(The actual phase diagram is much more complicated.)

Spaces of physical systems can have interesting and important homotopy types. We learn about them as children, when we learn a cartoon picture of {systems of water}, and define

$$\{\text{phases}\} = \pi_0\{\text{systems}\}.$$

Higher homotopy is also important. For example, topological insulators are interesting maps

$$BU(1) \times B\mathbb{Z}_2 = \mathbb{CP}^\infty \times \mathbb{RP}^\infty \to \{\text{quantum systems}\}.$$

What makes them useful in applications is that they are homotopically nontrivial: they are not homotopic to the constant map.

Phase classification, i.e. homotopy theory, of spaces of physical systems is also mathematically interesting.

Example: (0+1)d QFT = quantum mechanics (QM). It is fully mathematically rigorous: (super) Hilbert spaces $\mathcal{H}$ and (unbounded) self-adjoint operators $\hat{H}$.

$(\mathcal{H}, \hat{H})$ is compact if $\exp(-\tau\hat{H})$ is trace-class for $\tau > 0$. A supersymmetric QM model (SQM) is one equipped with a fermionic self-adjoint operator $\hat{Q}$ such that $\hat{Q}^2 = \hat{H}$ (up to conventions).

- $\{$compact QM models$\} \simeq *$
- $\{$noncompact SQM models$\} \simeq *$
- $\{$compact SQM models$\} \simeq \mathrm{K}$, the classifying space of the K-theory spectrum.

A complete mathematical definition of quantum field theory in high dimensions is far off. My work focuses on (1+1)d QFT.

{(1+1)d QFTs}	∞-dim space.	Optimistically, I expect a definition within $\sim$ 10 years.
$\cup$		
{(1+1)d CFTs}	finite-dim space.	Incomplete. Basic structure understood: (certain) pairs of Vertex Operator Algebras.
$\cup$		
{antiholo CFTs}	0-dim space.	Mathematically well-defined: VOAs with no nontriv irreps.

C = conformal. CFTs are the critical points for a function on {QFTs} called c, which is (hopefully) Morse–Bott. Its Morse flow is the Renormalization Group Flow. Low c = infrared.

Expectation: The space $\mathrm{SQFT} = \{$compact (1+1)d SQFTs$\}$ is homotopically interesting, just like $\mathrm{SQM} = \mathrm{K}$.

compact: Spectral constraint, like "$\exp(-\tau \hat{H})$ is trace-class."

S(upersymmetric): Fermionic operator $\hat{Q}$ such that $\hat{Q}^2 = \frac{\partial}{\partial \bar{z}}$, where $z, \bar{z} = t \pm x$ are the light cone coordinates on $\mathbb{R}^{1,1}$.

If you are an applied scientist: K-theory, because it classifies SQM models, has had enormous impact in the design of quantum materials. I expect there are finer SQFT-valued invariants.

How? A $(d+1)$-dimensional quantum material becomes a (0+1)d effective model by compactifying: treating all space directions as "small." If instead you compactify $(d-1)$ dimensions, you get a (1+1)d effective model.

Math example: Start with 6d $(2, 0)$ SCFT. Compactify on a 4-manifold. Gukov–Pei–Putrov–Vafa: top'l Vafa–Witten invariants.

The Witten index

Expectation: The space $\mathrm{SQFT} = \{$compact (1+1)d SQFTs$\}$ is homotopically interesting, just like $\mathrm{SQM} = \mathrm{K}$.

Question: Does SQFT even have multiple components?

Answer: Yes! The Witten index is a map $Z_{RR} : \pi_0\mathrm{SQFT} \to \mathrm{MF}_{\mathbb{Z}}$ = weak (pole at cusp) modular forms with integral q-expansion.

$$Z_{RR}(\mathcal{F})(\tau) := \int_{\text{fields } \phi\in\mathcal{F}} \exp\left(-\int_{E_\tau} \text{Lagrangian}(\phi)\right).$$

E_τ is the elliptic curve with complex structure τ, and nonbounding, aka RR, spin structure.

A priori, $Z_{RR}(\mathcal{F})(\tau, \bar{\tau})$ is a real-analytic weak modular form, i.e. a real-analytic function on the moduli space $\mathcal{M}$ of elliptic curves. (weak: pole at cusp.)

But formal arguments with path integrals give:

$$\frac{\partial}{\partial \bar{\tau}} Z_{RR}(\mathcal{F}) \propto \int_{\phi \in \mathcal{F}} (\hat{H} - \hat{P})\, e^{\int_{E_\tau} \mathsf{Lag}(\phi)} \propto \int_{\phi \in \mathcal{F}} \hat{Q}[\hat{Q}]\, e^{\int_{E_\tau} \mathsf{Lag}(\phi)}$$

where $\hat{H}$, $\hat{P}$, and $\hat{Q}^2$ are the energy, momentum, and supersymmetry operators, so that $\hat{H} - \hat{P} = \frac{\partial}{\partial \bar{z}} = \hat{Q}^2$.

$\hat{Q}$ acts like the de Rham d. In particular, if $\mathcal{F}$ is compact,

$$\int_{\phi \in \mathcal{F}} \hat{Q}[\hat{X}]\, e^{\int_{E_\tau} \mathsf{Lag}(\phi)} = 0$$

for any operator $\hat{X}$. This is a version of Stokes' theorem.

So $Z_{RR}(\mathcal{F})(\tau)$ is holomorphic, i.e. a weak modular form $\in \mathrm{MF}_{\mathbb{C}}$.

Moreover, the q-expansion of $Z_{RR}(\mathcal{F})(\tau)$ ends up counting (with signs) supersymmetric ground states in $\mathcal{F}$, because it is an index in K-theory. Thus $Z_{RR}(\mathcal{F})(\tau) \in \mathrm{MF}_{\mathbb{Z}}$. Since integers cannot deform, $Z_{RR}(\mathcal{F})$ is a deformation invariant.

Example: A string structure on a Riemannian manifold M is a spin structure together with a trivialization of the fractional Pontryagin class $\frac{p_1}{2}(T_M) \in \hat{\mathrm{H}}^4(M)$. Any string manifold determines a sigma model. (The string structure becomes the quantum B-field.)

Sigma models are not mathematically well-defined, but their taut-string limits are, and Z_{RR} becomes an integral over M which combines characteristic classes with Eisenstein series. Resulting $Z_{RR}(M)(\tau)$ is the Witten genus of M.

$$\{\text{string manifolds}\} \xrightarrow{\text{sigma model}} \mathrm{SQFT} \xrightarrow{Z_{RR}} \mathrm{MF}_{\mathbb{Z}}$$

$$\{\text{string manifolds}\} \xrightarrow{\text{Witten genus}} \mathrm{MF}_{\mathbb{Z}}$$

There is another object that also fits in the SQFT spot: the generalized cohomology theory TMF of topological modular forms. Just like $\mathrm{SQM} = \mathrm{K}$:

Conjecture (Witten, Segal, Stolz–Teichner): $\mathrm{SQFT} = \mathrm{TMF}$.

$\mathrm{TMF}^\bullet$ is a version of universal elliptic cohomology of Landweber–Ravenel–Stong. Witten discovered his genus while trying to understand the elliptic genus of Ochanine.

A proof of the conjecture (including a definition of (1+1)d QFT) would provide an analytic model for $\mathrm{TMF}^\bullet$. Currently its only construction requires hard derived algebraic geometry.

Physical theorem (Gaiotto–JF–Witten): SQFT carries an Ω-spectrum structure: it provides a cocycle model for a generalized cohomology theory.

Furthermore, cobordisms of string manifolds give homotopies of SQFTs: the map

$$\mathrm{MString} = \{\text{string manifolds}\}/\{\text{cobordism}\} \to \mathrm{SQFT}$$

is a map of generalized cohomology theories.

Remark: "zero" $\in \mathrm{SQFT}$ is any $\mathcal{F}$ in which supersymmetry is spontaneously broken. I will call such $\mathcal{F}$ null. The supersymmetry $\hat{Q}$ is like a differential (although $\hat{Q}^2 \neq 0$), and $\mathcal{F}$ is null when $\hat{Q}^2$ is exact. If $\mathcal{F} \sim$ zero, I will say it is nullhomotopic.

Beyond the Witten index

The mathematical Witten index $Z_{RR} : \pi_\bullet \mathrm{TMF} \to \mathrm{MF}_{\mathbb{Z}}$ is fully computed.

Method: There is a spectral sequence $\mathrm{H}^s(\mathcal{M}; L^w) \Rightarrow \pi_{2w-s}\mathrm{TMF}$, where $\mathcal{M}$ is the moduli space of elliptic curves, and L^w is the line bundle whose sections are weight-w modular forms.

$Z_{RR} : \pi_\bullet \mathrm{TMF} \to \mathrm{MF}_{\mathbb{Z}}$ is neither an injection nor a surjection.

Theorem (Bunke–Naumann): In topology, there is a secondary invariant, which sees beyond Z_{RR}.

What is its meaning physically? Why does it exist?

Recall the reason $Z_{RR}(\mathcal{F})$ was holomorphic:

$$\frac{\partial}{\partial \bar{\tau}} Z_{RR}(\mathcal{F}) \propto \int_{\phi \in \mathcal{F}} \mathrm{d}\hat{Q}\, e^{\int_{E_\tau} \mathsf{Lag}(\phi)} = 0 \text{ by Stokes' theorem.}$$

What if $\mathcal{F}$ is not compact? I.e. what if it has a "boundary" $\mathcal{S} = \partial\mathcal{F}$? Then $\mathcal{F}$ is not really a point in SQFT, but rather a nullhomotopy of $\mathcal{S} \in \mathrm{SQFT}$.

Physical Theorem (Gaiotto–JF): In this case, $Z_{RR}(\mathcal{F})(\tau, \bar{\tau})$ satisfies a holomorphic anomaly equation

$$\sqrt{-8\tau_2}\eta(\tau)\frac{\partial}{\partial \bar{\tau}} Z_{RR}(\mathcal{F}) = \int_{\phi \in \mathcal{S}} \hat{Q}\, e^{\int_{E_\tau} \mathsf{Lag}(\phi)} =: \langle \hat{Q} \rangle(\mathcal{S}).$$

Also, $f(\tau) := \lim_{\bar{\tau} \to -i\infty} Z_{RR}(\mathcal{F})(\tau, \bar{\tau}) \in \mathbb{Z}((q))$. I.e. $f(\tau)$ is an integral (generalized, weak) mock-modular form with shadow $\langle \hat{Q} \rangle(\mathcal{S})$.

Thm redux: If $\mathcal{S} = \partial \mathcal{F}$, then $\langle \hat{Q} \rangle(\mathcal{S})$ is a shadow of an integral mock-modular form.

Over $\mathbb{C}$ every modular form is a shadow. Over $\mathbb{Z}$ there may be an obstruction.

$$\text{obstruction}(\mathcal{S}) \in \frac{\mathbb{C}(\!(q)\!)}{\mathbb{Z}(\!(q)\!) + \mathrm{MF}_{\mathbb{C}}}.$$

Physical Theorem (Gaiotto–JF): This obstruction is a deformation invariant of $\mathcal{S}$, called the secondary Witten index.

Example:

Since $S^3 = \mathrm{SU}(2)$ is a Lie group, all of its characteristic classes vanish, and so it has a canonical string structure.

Topologists write "ν" for any class represented by this S^3.

Physical theorem (Gaiotto-JF-Witten): In the far infrared, the $\mathrm{SU}(2)$ sigma model is an antiholomorphic free fermion theory.

Direct calculation:

$$Z_{RR}(\nu) = 0, \qquad \langle \hat{Q} \rangle_{RR}(\nu) = \eta(\bar{\tau})^3,$$

$$\text{obstruction}(\nu) = \frac{1}{24} + \mathbb{Z}(\!(q)\!) + \mathrm{MF}_{\mathbb{C}} \neq 0.$$

So our mock-modularity invariant is nontrivial.

Take a $K3$ surface, and remove 24 points. The result can be given a string structure such that

$$\partial(K3 \smallsetminus 24\mathrm{pt}) = 24\nu.$$

Up to convention-dependent factors:

$$Z_{RR}(K3 \smallsetminus 24\mathrm{pt}) \propto q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots).$$

It is mock-modular with shadow $24\langle\hat{Q}\rangle(\nu) = 24\eta(\bar{\tau})^3$.

Corollary: $24\nu \simeq 0 \in \mathrm{SQFT}$.

Since Obstr$(\nu) = \frac{1}{24} \pmod{\mathbb{Z}}$, $\nu \in \pi_\bullet\mathrm{SQFT}$ has exact order 24.

Mathieu Moonshine

$$Z_{RR}(K3 \setminus 24\mathrm{pt}) \propto q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots).$$

Observation (Eguchi–Ooguri–Tachikawa): The coefficients are dimensions of irreps of the largest Mathieu group M_{24}.

M_{24} is a sporadic finite simple group. EOT observation is an analogue of McKay's Monstrous moonshine observation that

$$j(\tau) = \frac{E_4^3}{\Delta} - 744 = q^{-1}\Big(1 + (196883+1)q^2 + (21296876+196883+1)q^3 + (842609326+21296876+2\times196883+2)q^4 + \dots\Big)$$

are dimensions of irreps of the Monster sporadic group $\mathbb{M}$.

Theorem (Frenkel–Lepowski–Meurman): There exists an $\mathbb{M}$-equivariant holomorphic bosonic CFT whose Hilbert space $V = \bigoplus V_n$ has graded dimension $j(\tau)$ such that the characters $g \mapsto q^{-1}\sum_n \mathrm{tr}(g; V_n)q^n$ are all modular forms (for subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$). ($q^{-1}$ factor comes from the central charge.)

$$Z_{RR}(K3 \setminus 24\mathrm{pt}) \propto q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots).$$

Theorem (Gannon): This is the graded dimension of a graded M_{24}-module $V = \bigoplus V_n$ such that for each $g \in \mathrm{M}_{24}$, the character $g \mapsto q^{-1/8} \sum_n \mathrm{tr}(g; V_n) q^n$ is a mock modular form (for a specific subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$) with shadow $\mathrm{tr}(g; \mathrm{Perm}) \times \eta(\bar{\tau})^3$. ($\mathrm{Perm}$ is the standard permutation rep of M_{24}.)

Gannon's proof is number-theoretic. It does not tell much about M_{24}, and does not use $K3$, QFT, ...

Mathieu Moonshine Problem: Build this M_{24}-module as the Hilbert space of a (1+1)d SQFT.

Mathieu Moonshine Solution, first attempt:

M_{24} acts on $24\nu = \partial(K3 \smallsetminus 24\mathrm{pt})$ as the permutation module. If $24\nu \simeq 0$ M_{24}-equivariantly, then the corresponding nullhomotopy would give an SQFT whose Hilbert space has an M_{24}-action, with mock-modular characters and correct shadows.

In fact, it would suffice if this held in twisted-equivariant cohomology. Physicists call twistings 't Hooft anomalies.

Theorem (JF): 24ν is not twisted-M_{24}-equivariantly nullhomotopic, for any value of the twisting.

Proof: If it were, then it would also be M_{23}-equivariantly nullhomotopic, where $\mathrm{M}_{23} \subset \mathrm{M}_{24}$ is the second largest Mathieu group. Since $\mathrm{H}^\bullet(\mathrm{M}_{23}; \mathbb{Z})$ vanishes in degrees $\bullet \leq 5$, there is no anomaly. This means we can gauge the M_{23}-action, i.e. push forward along $\mathrm{M}_{23} \to \{1\}$. Result is $29\nu \not\simeq 0$.

Mathieu Moonshine Solution, second attempt:

The modular form Δ is not in the image of $Z_{RR} : \mathrm{TMF} \to \mathrm{MF}_{\mathbb{Z}}$. But 24Δ is. It is represented by a unique antiholomorphic SCFT discovered by Duncan. Its automorphism group is the largest Conway group Co_1, another sporadic simple group.

Nonequivariantly, $24\Delta \times \nu = 0 \in \pi_\bullet \mathrm{TMF}$.

Conjecture: $24\Delta \in \pi_\bullet \mathrm{TMF}$ has a twisted-Co_1-equivariant refinement. (**JF–Treumann:** value of the twisting.)

Conjecture: $24\Delta \times \nu$ is not nullhomotopic Co_1-equivariantly, but it is nullhomotopic M_{24}-equivariantly. **Note:** $\mathrm{M}_{24} \subset \mathrm{Co}_1$.

Theorem (JF): The twistings and shadows match: up to an overall normalization, $Z_{RR}(\mathcal{F})$ will have the same mock-modularity as predicted in generalized Mathieu Moonshine.

To call something moonshine, you should have a version of the genus-zero property. In Monstrous Moonshine, this is the statement that each character defines an isomorphism (upper half plane)/$\Gamma \overset{\sim}{\to} \mathbb{CP}^1$ for some $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$.

Theorem (Cheng–Duncan): This is equivalent to an optimal growth condition on the behaviour of the characters near cusps. Optimal growth makes sense for mock-modular forms.

Pre-theorem (JF): The optimal growth condition in Mathieu Moonshine is equivalent to saying that $24\Delta \times \nu$ is nullhomotopic among M_{24}-equivariant topological cusp forms Tcf.

Remark: Not yet clear which physics leads to strong modular forms (regular at $\tau = i\infty$) or to cusp forms (vanish at $\tau = i\infty$).

Remark: Non-topologically, $\mathrm{cf} = \mathrm{mf}\Delta \cong \mathrm{mf}$. But Δ is not a topological modular form, and $\mathrm{Tmf} \ncong \mathrm{Tcf}$.

Conjecture redux: $24\Delta \times \nu \simeq 0$ in M_{24}-equivariant TMF.

Theorem (JF): The appropriate Borel-equivariant $\mathrm{Tmf}[\frac{1}{2}]$-cohomology group vanishes.

Borel-equivariant cohomology approximates genuinely-equivariant cohomology by replacing stacks with their classifying spaces.

Expect that Borel-equivariant is a power series completion of genuinely-equivariant. (Compare: Atiyah–Segal completion in K-theory.) So theorem $\Rightarrow$ conjecture perturbatively ($p \neq 2$).

Method: Direct calculation with Atiyah–Hirzebruch spectral sequences, Steenrod operators, etc.

Direct p=2 calculation is too hard: we do not even know the ordinary cohomology of M_{24} at the prime p=2.

Thank you!

Further details:

[arXiv:1811.00589] Holomorphic SCFTs with small index

[arXiv:1902.10249] A note on some minimally supersymmetric models in two dimensions

[arXiv:1904.05788] Mock modularity and a secondary elliptic genus

[arXiv:2006.02922] Topological Mathieu moonshine

[these slides] http://categorified.net/Dal-SQFTs.pdf

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