FIRM CATEGORIES — EXERCISES

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These exercises are based on joint work in progress with Jacob McNamara and David Reutter.

Throughout, $\mathcal{V}$ is some fixed presentably symmetric category in which I will enrich my categories. For example, $\mathcal{V}$ can be Set. I will mostly suppress $\mathcal{V}$ from the notation — for example, $\mathsf{Pr} := \mathsf{Pr}_{\mathcal{V}}$. Throughout, "category" will mean "valent $\mathcal{V}$-category": I do not impose a univalence (aka Rezk-completeness) condition. For C a ($\mathcal{V}$-)category, with objects $x, y \in C$, write $C(x,y) \in \mathcal{V}$ for $\hom_C(x,y)$.

(1) Let C be a nonunital category. Write C_+ for the unital category that freely adjoins units: in other words, $(-)_+$ is the left adjoint for the forgetful functor $\{\text{unital categories}\} \to \{\text{nonunital categories}\}$. By adjunction, any nonunital functor $f : C \to \mathcal{V}$ enhances to a unital functor $f_+ : C_+ \to \mathcal{V}$. Show that for any functor $f : C \otimes C^{\mathrm{op}} \to \mathcal{V}$, $\mathrm{coend}_C f = \mathrm{coend}_{C_+} f_+$.

(2) Let C be a nonunital $\mathcal{V}$-category. A presheaf $f : C^{\mathrm{op}} \to \mathcal{V}$ is *firm* if $\mathrm{coend}_{y \in C} C(x,y) f(y) \to f(x)$ is a isomorphism (for all x). The category C is firm if for every y, the representable presheaf $C(-,y)$ is firm. Show that C is firm iff C^{op} is firm.

(3) Write $\mathrm{P}(C)$ for the category of firm presheaves on C. Show that if C happens to be unital, then $\mathrm{P}(C)$ is the category of unital presheaves on C — presheaves in the conventional sense — so that the notation is not conflicting.

(4) Let C, D be firm categories. A *firm profunctor* $C \nrightarrow D$ is a firm presheaf on $C^{\mathrm{op}} \otimes D$. Show that firm profunctors compose. Conclude that firm profunctors $C \nrightarrow D$ induce colimit-preserving functors $\mathrm{P}(C) \to \mathrm{P}(D)$.

(5) Prove that $\mathrm{P}(C)$ is a retract (in the strong sense) of $\mathrm{P}(C_+)$. Conclude that $\mathrm{P}(C)$ is presentable.

(6) Prove that $\mathrm{P}(C)$ is 1-dualizable in Pr with dual $\mathrm{P}(C^{\mathrm{op}})$.

(7) Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{Pr}$. Note that Pr is closed-monoidal, so that $\mathsf{Pr}(\mathcal{A}, \mathcal{B}) \in \mathsf{Pr}$ and the composition law $\mathsf{Pr}(\mathcal{B}, \mathcal{C}) \times \mathsf{Pr}(\mathcal{A}, \mathcal{B}) \to \mathsf{Pr}(\mathcal{A}, \mathcal{C})$ enhances to a Pr-map $\circ : \mathsf{Pr}(\mathcal{B}, \mathcal{C}) \otimes \mathsf{Pr}(\mathcal{A}, \mathcal{B}) \to \mathsf{Pr}(\mathcal{A}, \mathcal{C})$, and hence the composition functors $\circ f$ and $f \circ$ have right adjoints. Given $f \in \mathsf{Pr}(\mathcal{A}, \mathcal{B})$, its *left* and *right duals* are respectively:

$$
\begin{aligned}
{}^{\vee}f &:= (\circ f)^R(\mathrm{id}_{\mathcal{A}}) \in \mathsf{Pr}(\mathcal{B}, \mathcal{A}),\\
f^{\vee} &:= (f \circ)^R(\mathrm{id}_{\mathcal{B}}) \in \mathsf{Pr}(\mathcal{B}, \mathcal{A}).
\end{aligned}
$$

In other words, these are the universal morphisms in Pr equipped with natural transformations ${}^{\vee}f f \Rightarrow \mathrm{id}_{\mathcal{A}}$ and $f f^{\vee} \Rightarrow \mathrm{id}_{\mathcal{B}}$.

Show that for $f \in \mathsf{Pr}(\mathcal{A}, \mathcal{B})$, the following are equivalent:

(a) f is an internal right adjoint in Pr.
(b) The right adjoint functor $f^R \in \mathsf{CAT}(\mathcal{B}, \mathcal{A})$ is a Pr-morphism.
(c) The canonical comparison map $f^{\vee} \Rightarrow f^R$ in $\mathsf{CAT}(\mathcal{B}, \mathcal{A})$ is an isomorphism.
(d) The universal map $f f^{\vee} \Rightarrow \mathrm{id}_{\mathcal{B}}$ is the evaluation morphism in an internal adjunction $f \dashv f^{\vee}$.

   (e) For any $g \in \mathsf{Pr}(\mathcal{C},\mathcal{B})$ and $h \in \mathsf{Pr}(\mathcal{C},\mathcal{A})$, the comparison map $\mathsf{Pr}(\mathcal{C},\mathcal{A})(h, f^\vee g) \to \mathsf{Pr}(\mathcal{C},\mathcal{B})(fh, g)$ [whisker with f and then compose with the canonical map $ff^\vee \Rightarrow \mathrm{id}_{\mathcal{B}}$] is an isomorphism.
   (f) For any $g \in \mathsf{Pr}(\mathcal{A},\mathcal{B})$, the comparison map $\mathsf{Pr}(\mathcal{A},\mathcal{A})(\mathrm{id}_{\mathcal{A}}, f^\vee g) \to \mathsf{Pr}(\mathcal{A},\mathcal{B})(f, g)$ is an isomorphism.
   (g) The comparison map $\mathsf{Pr}(\mathcal{A},\mathcal{A})(\mathrm{id}_{\mathcal{A}}, f^\vee f) \to \mathsf{Pr}(\mathcal{A},\mathcal{B})(f, f)$ is an isomorphism.
   (h) The comparison map $\mathsf{Pr}(\mathcal{A},\mathcal{A})(\mathrm{id}_{\mathcal{A}}, f^\vee f) \to \mathsf{Pr}(\mathcal{A},\mathcal{B})(f, f)$ has id_f in its image.

(8) Let $\mathcal{C} \in \mathsf{Pr}$ and identify objects $c \in \mathcal{C}$ with Pr-morphisms $c : \mathcal{V} \to \mathcal{C}$. The space of *atomic morphisms* $\mathcal{C}^{\mathrm{at}}(c,d)$ are by definition $c^\vee(d) \in \mathcal{V}$. Show that there is a natural transformation $\mathcal{C}^{\mathrm{at}}(-,-) \to \mathcal{C}(-,-)$. Use the universal property to show that $\mathcal{C}^{\mathrm{at}}(-,-)$ has a nonunital composition law, and that the natural transformation $\mathcal{C}^{\mathrm{at}}(-,-) \to \mathcal{C}(-,-)$ is a nonunital functor. Conclude that every presentable category has a nonunital category (on the same objects) canonically associated do it.

(9) Which morphisms $f \in \mathsf{Pr}(\mathcal{C},\mathcal{D})$ induce nonunital functors $\mathcal{C}^{\mathrm{at}} \to \mathcal{D}^{\mathrm{at}}$?

(10) Conclude from exercise 7 that the following are equivalent for an object $c \in \mathcal{C}$:
   (a) c is *atomic*: the functor $\mathcal{C}(c,-) : \mathcal{C} \to \mathcal{V}$ preserves colimits.
   (b) The comparison map $\mathcal{C}^{\mathrm{at}}(c,-) \to \mathcal{C}(c,-)$ is an isomorphism.
   (c) The comparison map $\mathcal{C}^{\mathrm{at}}(c,c) \to \mathcal{C}(c,c)$ has id_c in its image.

(11) Set $\mathcal{C} = \mathcal{V}$ to be your favourite presentable category of topological vector spaces. Who is $\mathcal{V}^{\mathrm{at}}$? Justify the use of the word “atomic.” [This is historically not the reason that “atomic” was used, but it is a happy accident.]

(12) Prove the *firm Yoneda lemma*: if C is a firm category and $f \in \mathrm{P}(C)$ is any firm presheaf on C, then
$$f(c) = \mathrm{P}(C)^{\mathrm{at}}(C(-,c), f).$$
In particular, the nonunital category C can be recovered from the unital category $\mathrm{P}(C)$ (together with its marking by representables):
$$C(c,d) = \mathrm{P}(C)^{\mathrm{at}}(C(-,c), C(-,d)).$$