Higher Algebraic Closure

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joint work in progress w/ D. Reutter

slides: categorified.net/Feza.pdf

Nullstellensätze and Tannakian correspondences

Theorem [Hilbert, Gauss, Galois] If A is a nonzero commutative $\mathbb{R}$-algebra satisfying a size constraint, then there exists a com alg homomorphism $A \to \mathbb{C}$, the alg of complex numbers.
Moreover, under a stronger size constraint, the canonical map

$$A \longrightarrow \hom_{sets}(\hom_{alg}(A, \mathbb{C}), \mathbb{C})$$

is an iso.

Theorem [Deligne] If $\mathcal{A}$ is a non-zero sym mon category $/\mathbb{R}$ satisfying a size constraint, then there exists a sym mon functor $\mathcal{A} \longrightarrow \mathrm{sVec}_{\mathbb{C}}$, the category of super vector spaces. Moreover, under a stronger size constraint, $\mathcal{A} \xrightarrow{\sim} \hom_{\mathrm{spoid}}(\hom_{\otimes}(\mathcal{A}, \mathrm{sVec}), \mathrm{sVec})$

Goal of talk Higher category of complex super duper ...

Higher Commutative algebras

[scheimbauer]

The **based loops** in a pointed n-category $\mathcal{C}^n \ni 1$ is the $(n-1)$-category $\Omega\mathcal{C}^n := \mathrm{End}_{\mathcal{C}^n}(1)$. A **tower** is a categorical Ω-spectrum: a sequence $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2, \ldots$ with equivalences

$$\Omega\mathcal{C}^n \simeq \mathcal{C}^{n-1}$$

Infinite loop structure $\rightsquigarrow$ all $\mathcal{C}^n$ are symmetric monoidal.

Examples
- $\mathbb{K}$, $\mathrm{Rep}_{\mathbb{K}}(G)$, $\{\mathbb{K}\text{-linear categories w/ } G\text{-action}\}$, $\{\mathbb{K}\text{-linear 2-categories w/ } G\text{-action}\}$, ...
- any sym $\otimes$ n-cat $\mathcal{C}$ gives a tower $B^{\bullet-n}\mathcal{C}$ w/ $(B^{\bullet-n}\mathcal{C})^m = \begin{cases} B^{m-n}\mathcal{C}, & m \geq n \\ \Omega^{n-m}\mathcal{C}, & n \geq m \end{cases}$ [one-pt delooping]

Higher semisimplicity

[Gaiotto–JF]

A $\mathbb{K}$-linear n-category is **semisimple** if it is:

- **Karoubean**: closed under $\oplus$'s and splitting of n-idempotents
- **rigid**: all $(<n)$-morphisms have adjoints
- **locally semisimple**: All endomorphism rings of $(n-1)$-morphisms are semisimple.

Example If $\mathcal{C}$ is a semisimple monoidal $(n-1)$-cat, then $\Sigma\mathcal{C} :=$ Karoubi completion of $B\mathcal{C}$ is a semisimple n-cat.

$\hookrightarrow \cong$ {f.g. proj. $\mathcal{C}$-modules}

$\cong$ {separable algebras and bimodules in $\mathcal{C}$}

I will take semisimple towers as my higher com algs.

I don't know what a non-s.s. "abelian n-category" is.

Higher Schur Theory

[Douglas - Reutter]

A morphism $X \xrightarrow{f} Y$ in an n-cat is **monic** if $\forall Z$, the induced functor $\hom(Z,X) \xrightarrow{f_\circ} \hom(Z,Y)$ is faithful. An object X is **simple** if every **non-zero** $X \xrightarrow{f} Y$ is monic. In a s.s. n-cat, every object is a $\oplus$ of simples.

Warning: When $n \geq 2$, the usual Schur's lemma, that **nonzero maps between simples are isos**, fails! Yet:

Higher Schur's Lemma: In a s.s. n-cat $\mathcal{C}$, "connected by a non-zero map" is an equiv rel'n on the simple objects.

$$\pi_0 \mathcal{C} := \text{equivalence classes} = \text{Schur components}$$

$\pi_0 \mathcal{C}$ indexes the simple summands (blocks) of $\mathcal{C} \in n\mathrm{Cat}$.

Abelian extensions

Strategy: If $\mathcal{W}^\bullet$ is an alg closed tower, then $\mathcal{W}^n$ is alg closed among n-cats. Pf: $\Sigma^{\bullet - n}: \{\text{sym} \otimes n\text{-cats}\} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \{\text{towers}\}: \Omega^{\infty - n}$.

So build $\mathcal{W}^n$ as an extension of $\Sigma \mathcal{W}^{n-1}$!

simple summands $= \pi_0 \mathcal{W}^n$

Example:

- $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\sqrt{-1}$ is a $\mathbb{Z}/2$-graded extension.
- $s\mathrm{Vec}_{\mathbb{C}} = \underset{= \Sigma\mathbb{C}}{\mathrm{Vec}_{\mathbb{C}}} \oplus \mathrm{Vec}_{\mathbb{C}} \cdot (\text{fermion})$ is a $\mathbb{Z}/2$-graded extension.

Proposition [JF-Yu]: Suppose $\mathcal{C}^n$ is rigid monoidal semisimple n-cat w/ $\Omega^{n-1}\mathcal{C}^n = s\mathrm{Vec}$. Then $\pi_0 \mathcal{C}^n$ is a group, and $\mathcal{C}^n$ is a $(\pi_0 \mathcal{C}^n)$-graded extension of $\Sigma \Omega \mathcal{C}^n$.

connected comp of 1.

Other words, the "field extension" $\mathcal{W}^n \to \mathcal{W}^{n+1}$ is "abelian".
We think that in fact $\mathbb{C} \to \mathcal{W}^\bullet$ is an abelian.

Invertibles

If $\mathcal{C}^\bullet$ is a tower, then $(\mathcal{C}^\bullet)^\times = \{$invertible objects, isos, in $\mathcal{C}^\bullet\}$ is a Ω-spectrum of spaces.

Extension theory: Abelian extensions of $\mathcal{C}^n$ are classified by maps $A \longrightarrow (\Sigma\mathcal{C})^\times$. Care about case $\mathcal{C}^n = \Sigma \mathcal{W}^{n-1}$.

Strategy: If $\mathcal{W}^\bullet$ is alg closed, then $\mathcal{W}^\times = I\mathbb{C}^\times$ is the Cartier dual to the sphere spectrum i.e. $[T, I\mathbb{C}^\times] = [\pi_0 T, \mathbb{C}^\times]$ for any spectrum T. Pf: Test univ. prop of $\mathcal{W}$ against "gp algs".

By induction, $(\Sigma^2 \mathcal{W}^{n-1})^\times$ and $I\mathbb{C}^\times[n+1]$ agree in degs ≥ 2.

Using universal prop of $I\mathbb{C}^\times$ and that $\mathbb{Z}$ has global dim $= 1$:

Main Theorem: There exists a unique s.s. tower $\mathcal{W}^\bullet$ w/ $\mathcal{W}^1 = \mathrm{sVec}_{\mathbb{C}}$ and $\mathcal{W}^\times = I\mathbb{C}^\times$. This $\mathcal{W}^\bullet$ is alg. closed.

Universal target categories

The existence of a tower $\mathcal{W}^\bullet$ w/ $\mathcal{W}^1 = \mathrm{sVec}$ and $\mathcal{W}^\times = I\mathbb{C}^\times$ was first speculated by Hopkins and appears in print in Freed–Hopkins, Reflection Positivity. Their motivation: find a "universal" category where "all" topological field theories take values. For example, the universal prop of $I\mathbb{C}^\times$ says it is the correct target for invertible tfts if you want tfts to be determined by their partition functions.

Our $\mathcal{W}^\bullet$ replaces invertible w/ semisimple: every s.s. tft takes values in $\mathcal{W}$, and is determined by its partition fn.

category of extended operators is semisimple.

cat of extended ops is trivial.

Galois/Tannaka philosophy: tfts in $\mathcal{V}^\bullet$ = tfts in $\mathcal{W}^\bullet$ enhanced with $\mathrm{Gal}(\mathcal{W}^\bullet/\mathcal{V}^\bullet)$ symmetry.

Computing the higher absolute Galois Group

Gal(ab ext) = dual to grading gp. So need to compute $\pi_0 \mathcal{W}^n$.

This is:

$$\{\text{s.s. } n\text{D TFTs}\} \Big/ \{\text{non-zero interfaces}\}$$

equivalently: TFTs of Levin-Wen type.

more casually: $\pi_0 \mathcal{W}^n = \{\text{s.s. TFTs}\} / \{\text{TFTs w/ top'l b.c.}\}$

Lan-Kong-Wen Surgery: In an nD tft, the operators of $\dim \leq \frac{n}{2}-1$ form a sym mon subsector. By universality of $\mathcal{W}$, they can be identified w/ $\mathrm{Rep}(\mathcal{G})$ for some "gauge gp" $\mathcal{G}$.

Ungauge / condense / surger $\mathcal{G}$: Get new tft w/o ops of $\dim \leq \frac{n}{2}-1$

Remote Detectability / Poincare duality: Also no ops of $\dim \geq \frac{n}{2}$.

But: No operators at all $\iff$ invertible.

Surgery results

related by a top'l interface

In $(2k)$D, every TFTs is $\overbrace{\text{cobordant}}$ to an invertible one.

In $(2k+1)$D, if $2k+1 \geq 5$, the only obstruction is a class

$$\ell^{2k+2} := \frac{\{\text{finite ab gps w/ a nondeg } (\text{skew})^{k+1}\text{-sym form}\}}{\text{lagrangian correspondences}}$$

$$\ell^n \cong \begin{cases} 0, & n \text{ odd} \\ \mathbb{Z}/2, & n/2 \text{ odd} \\ \mathbb{Z}/2 \oplus \bigoplus_{p=1(4)} (\mathbb{Z}/2)^2 \oplus \bigoplus_{p=3(4)} \mathbb{Z}/4, & n/2 \text{ even} \end{cases}$$

Theorem: When $n \geq 4$, there is a LKW surgery LES

$$\cdots \to \pi_n \mathrm{Gal}(\mathcal{W}^\bullet/\mathbb{C}) \longrightarrow \pi_n \mathcal{S} \longrightarrow \hom(\ell^n, \mathbb{C}^\times) \longrightarrow \pi_{n-1}\mathrm{Gal} \to \cdots$$

Surgery results

Theorem: When $n \geq 4$, there is a LKW surgery LES

$$\cdots \to \pi_n \mathrm{Gal}(\mathcal{W}^\circ/_{\mathbb{C}}) \to \pi_n \mathbb{S} \xrightarrow{\star} \hom(\ell^n, \mathbb{C}^\times) \to \pi_{n-1}\mathrm{Gal} \to \cdots$$

But $\star$ is usually zero:

- $\ell^n = 0$ if n is odd
- if $n = 0\,(4)$, then $\exists$ bosonic refinements of all classes in ℓ^n.
- if $n = 2\,(4)$, then $\ell^n = \mathbb{Z}/_2$ is an Arf-Kervaire invariant,

  and Hill-Hopkins-Ravenel show $\star = 0$ except $n = 2, 6, 14, 30, 62,$ and maybe 126.

Corollary: An nD QFT w/ a nontrivial grav. anomaly is necessarily gapless unless $n = 1, 5, 13, 29, 61,$ or maybe 125.

Comparison w/ Classical Surgery

Classical surgery obstructs manifolds from being cobordant to speres.
It uses $L(\mathbb{Z})$, or better L(H-finite spectra w/ $\mathbb{S}$-valued quad. forms).
It relates L, $\mathbb{S}$, and PL = piecewise-linear autos of $\mathbb{R}^n$.

Our surgery looks like the same thing, but with

$$\ell = L(\pi\text{-finite spectra w/ } I\mathbb{C}^\times\text{-valued quad. forms}).$$

This looks a lot like a profinite completion of L.

Thus, it looks like $Gal \simeq$ profinite completion of PL.

This is still very much in progress.

Comparison to MTCs, RT thy

Surgery thy, both classical and our versions, breaks down around $\dim \approx 3, 4$ because of nonabelian knottedness.

Instead of ℓ^4, the gp in the LES is

$$\text{super quantum Witt gp} = \text{super MTCs} \,/\, \text{Drinfeld centres}$$

This is known to be a non-canonically split extension

$$\text{super } q\text{Witt} \cong \ell^4 \oplus (\mathbb{Z}/2)^{\oplus\infty} \oplus \mathbb{Z}^{\oplus\infty}.$$

It classifies cobordism classes of Reshetikhin–Turaev theories.

The universal 3-cat $\mathcal{W}^3$ of RT thys was first built by Freed–Scheimbauer–Teleman. Their work inspired ours.

Speculations and closing comments

- If indeed there is a coherent surgery

$$\mathrm{Gal}(\mathcal{W}^\circ/\mathbb{C}) \to GL_1(\$) \longrightarrow GL_1(\ell^\vee)$$

then almost surely $\mathrm{Gal}(\mathcal{W}^\circ/\mathbb{C})$ will be abelian (i.e. E_∞)
so that $\mathcal{W}^\circ$ is a cyclotomic extension of $\mathbb{C}$.

- $\pi_0\,\mathrm{Gal}(\mathcal{W}^\circ/\mathbb{C}) = *$, but $\pi_0 PL = \mathbb{Z}/2 = \pi_0\,\mathrm{Gal}(\mathcal{W}^\circ/\mathbb{R})$.

To really build a comparison map $PL \to \mathrm{Gal}(\mathcal{W}^\circ/\mathbb{R})$
probably need some notion of reflection positivity.

- In positive characteristic, construction of s.s. alg closure
still basically works. Extension $\mathbb{k} \to \mathcal{W}^\circ$ is not Galois
and not G-graded in deg $=1$: $\mathcal{W}' = \mathrm{Ver}_p$ [Ostrik]

$\mathcal{W}^\circ$ is a purely insep extension of its max sep subext,
and the Galois gp still looks $\approx$ like PL.