How to Build a Hopf Algebra

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SCGCS Annual Meeting, 20 Nov 2025, Simons Foundation

Based on arXiv:2508.16787, joint with David Reutter

These slides:
https://categorified.net/Hopf-SimonsFoundation.pdf

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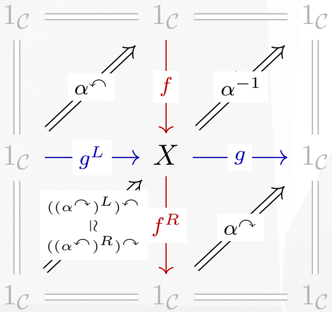
Main Theorem (JF–Reutter): Let $(\mathcal{C}, 1_\mathcal{C} \in \mathcal{C})$ be a pointed $(\infty,3)$-category. Given a retract $\left(1_\mathcal{C} \xrightarrow{f} X \xrightarrow{g} 1_\mathcal{C},\ \alpha : \mathrm{id}_{1_\mathcal{C}} \overset{\sim}{\Rightarrow} gf\right)$, if f is a left adjoint, g is a right adjoint, and the canonical 2-morphism $\alpha^{\curvearrowright} : f^R \Rightarrow g$ is a right adjoint (equiv. canonical 2-morphism $\alpha^{\curvearrowleft} : g^L \Rightarrow f$ is a left adjoint), then

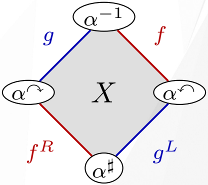


$$\in \Omega^2\mathcal{C} := \mathrm{End}_{\mathrm{End}_\mathcal{C}(1_\mathcal{C})}(\mathrm{id}_{1_\mathcal{C}}).$$

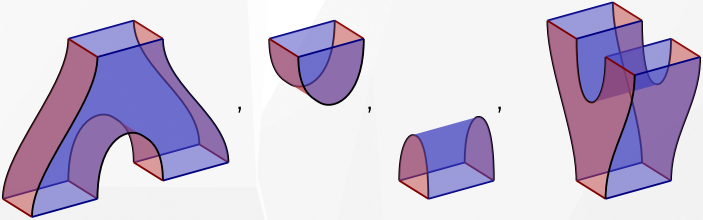
is a bialgebra, and its opposite bialgebra is Hopf (admits an antipode which is not necessarily invertible).

Incoherent topological unpacking of Main Theorem

Poincaré-dually, the composite is a square with alternating boundary conditions:



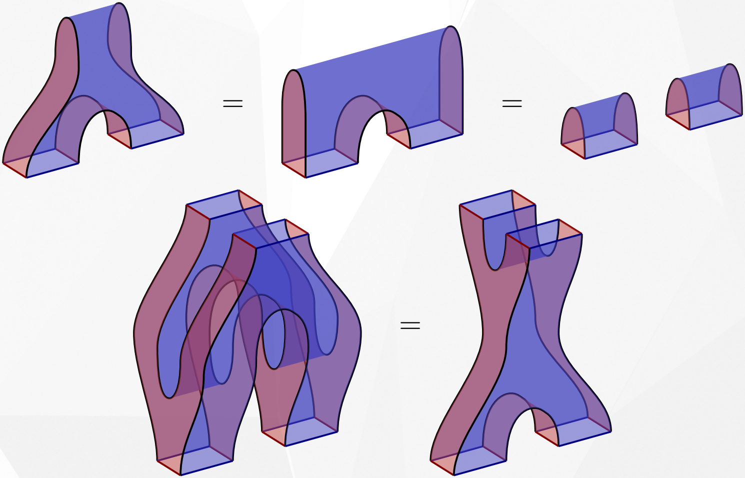
The multiplication, unit, counit, and comultiplication are:



Compare: Reutter 2017, Freed–Teleman 2022, Dimofte–Niu 2024.

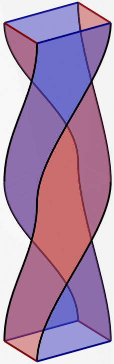
Incoherent topological description of Hopf axioms

Graphically, the retract axiom $\mathrm{id}_{1_\mathcal{C}} \overset{\sim}{\Rightarrow} gf$ says that bigons, and bigonal holes, can be created and destroyed.



Dangers of incoherent topological unpacking

- Tracking framing data is annoying.
- Calculus of defects not rigorously established.
- Bialgebra axioms hold up to homotopy for top'l reasons. Want them ∞-coherently.
- Want $\mathbf{Ret}^{\mathrm{adj}}(\mathcal{C}) \to \mathbf{Hopf}(\Omega^2\mathcal{C})$ as a functor.
- We do not require full adjunctibility. Without full adjunctibility, cannot produce the antipode as a bordism.
- This is a feature: we can get every Hopf algebra in every presentably braided monoidal $(\infty, 1)$-category, including ones with noninvertible antipode.



What the antipode wants to be, but isn't without full adjunctibility.

Inductive definition:

$$\mathbf{Cat}_{(\infty,n)} := \{\text{categories enriched in } \mathbf{Cat}_{(\infty,n-1)}\}.$$

$\mathbf{Cat}_{(\infty,\infty)}$ is the $n \to \infty$ limit. Technically:

$$\mathbf{Cat}_{(\infty,\infty)} := \mathbf{Sheaves}\left(\bigcup_{n\to\infty} \mathbf{Cat}_{(\infty,n)}\right).$$

Picture (∞, ∞)-categories as oriented cell complexes, with a k-cell for each generating k-morphism, oriented from source to target.

The lax tensor product $\otimes$

Want to orient a product of oriented cell complexes. I.e. given cells $a \in A$, $b \in B$, which direction points $a \otimes b \in A \otimes B$? Strategy: Oriented cell complex $A \rightsquigarrow$ chain complex $C_*(A)$, via $\partial a = \text{target}(a) - \text{source}(a)$; ask $C_*(A \otimes B) = C_*(A) \otimes C_*(B)$, of course with $\partial(a \otimes b) = \partial a \otimes b + (-1)^{\dim a} a \otimes \partial b$.

Theorem (Campion): $\mathbf{Cat}_{(\infty,\infty)}$ admits a unique closed monoidal structure $\otimes$ compatible with this strategy.

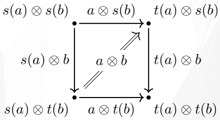
This $\otimes$ is not symmetric: $A \otimes B \not\cong B \otimes A$ in general. Rather, $B \otimes A = (A^{\mathrm{op}} \otimes B^{\mathrm{op}})^{\mathrm{op}}$, where $(-)^{\mathrm{op}}$ takes opposites in all odd directions.

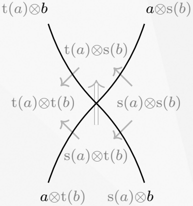
If $A \in \mathbf{Cat}_{(\infty,m)}$ and $B \in \mathbf{Cat}_{(\infty,n)}$, then $A \otimes B \in \mathbf{Cat}_{(\infty,m+n)}$.

Pictures of $\otimes$

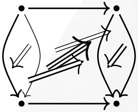
Pasting diagram | Poincaré dual

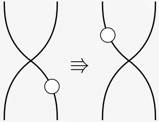
$(s(a) \xrightarrow{a} t(a)) \otimes (s(b) \xrightarrow{b} t(b))$



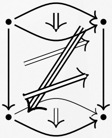


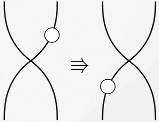
arrow $\otimes$ 2-cell





2-cell $\otimes$ arrow

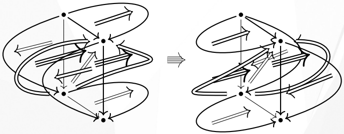


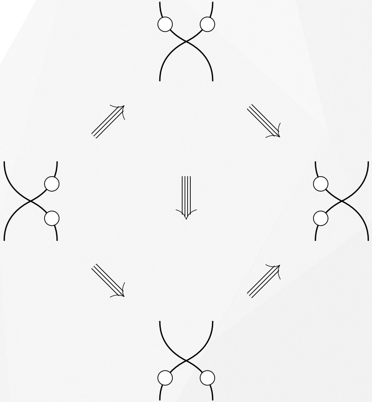


(strictly speaking. . .)

More pictures of $\otimes$

2-cell $\otimes$ 2-cell





Lax and oplax natural transformations

If $f, g : A \to B$ are functors of (∞, ∞)-categories, a strong natural transformation $\eta : f \Rightarrow g$ chooses a morphism $\eta a : fa \to ga$ for each object $a \in A$, and for each $\alpha : a \to a'$, a naturality isomorphism $(g\alpha) \circ (\eta a) \cong (\eta a') \circ (f\alpha)$. The idea of (op)lax natural transformations is to allow noninvertible naturality morphisms. $\mathrm{Fun}^{(\mathrm{op})\mathrm{lax}}(X, Y) :=$ functors $X \to Y$ and (op)lax natural transfors. These are defined by hom-tensor adjunction:

$$\mathrm{maps}(W \to \mathrm{Fun}^{\mathrm{lax}}(X, Y)) = \mathrm{maps}(X \otimes W \to Y)$$
$$\mathrm{maps}(W \to \mathrm{Fun}^{\mathrm{oplax}}(X, Y)) = \mathrm{maps}(W \otimes X \to Y).$$

Further ingredients: localization, $\oslash$, and Ω

$L_n : \mathbf{Cat}_{(\infty,\infty)} \to \mathbf{Cat}_{(\infty,n)}$, the left adjoint to the inclusion $\mathbf{Cat}_{(\infty,n)} \subset \mathbf{Cat}_{(\infty,\infty)}$, inverts all $(> n)$-morphisms.

$\mathbf{Cat}^*_{(\infty,\infty)}$:= {categories equipped with a distinguished object}

The smash product of $(X \ni 1_X), (Y \ni 1_Y) \in \mathbf{Cat}^*_{(\infty,\infty)}$ is

$$X \oslash Y := \frac{X \otimes Y}{X \otimes \{1_Y\} \underset{\{1_X\}\otimes\{1_Y\}}{\sqcup} \{1_X\} \otimes Y}.$$

$\oslash$ is hom-tensor adjoint to strongly pointed functors — $f : X \to Y$ with $f1_X \cong 1_Y$ — and their (op)lax natural transformations.

$\vec{S}^1 := \mathrm{B}\mathbb{N}$ is the directed circle. The based loop category of $X \in \mathbf{Cat}^*_{(\infty,\infty)}$ is $\Omega X := \mathrm{Fun}_*^{\mathrm{oplax}}(\vec{S}^1, X) = \mathrm{End}_X(\text{basepoint})$.

Monad $\oslash$ Monad = Bialgebra

A braided monoidal $(\infty, 1)$-category is a monoid object in $\{$monoidal $(\infty, 1)$-categories$\}$. If $\mathcal{C} \in \mathbf{Cat}^*_{(\infty,3)}$, then $\Omega\mathcal{C}$ is a mon. $(\infty, 2)$-cat. and $\Omega^2\mathcal{C}$ is a braided mon. $(\infty, 1)$-cat.

Vague idea: (op)lax monoidal functor is some type of (co)algebra.

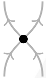
$\mathrm{Mnd} := \mathrm{B}$(free monoidal 1-category on an associative algebra). In other words, $\mathrm{maps}_*(\mathrm{Mnd}, X) = \mathrm{ob}\,\mathrm{Alg}(\Omega X)$. Define oplax algebra morphisms by $\mathrm{Alg}^{\mathrm{oplax}}(\Omega X) := \mathrm{Fun}_*^{\mathrm{oplax}}(\mathrm{Mnd}, X)$.

Theorem (Hadzihasanovic, JF–R): If $X \in \mathbf{Cat}^*_{(\infty,3)}$, then $\mathrm{Alg}^{\mathrm{oplax}}(\mathrm{Alg}^{\mathrm{oplax}}(\Omega X)) = \mathrm{BiAlg}(\Omega^2 X)$. Equivalently:

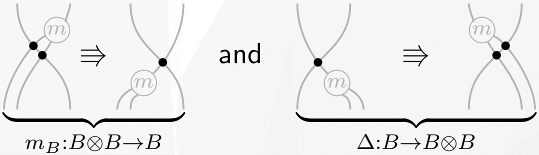
$$L_3(\mathrm{Mnd} \oslash \mathrm{Mnd}) = \mathrm{B}^2(\text{free braided 1-category on a bialgebra}).$$

Incoherent pictures of Mnd $\circledwedge$ Mnd = Bialg

Can think of $\mathrm{Arr} \otimes \mathrm{Arr}$ as a pasting square. Poincaré dually, it is two wires crossing. Writing in grey the cells that are smashed out, here is a Poincaré dual picture of the generating cell in $\vec{S}^1 \circledwedge \vec{S}^1$:

$B =$ 

Mnd is generated by $\vec{S}^1$ and a multiplication 2-cell m (and a unit, omitted). Multiplying m with the generating 1-cell in Mnd gives the two generating 3-cells in $\mathrm{Mnd} \circledwedge \mathrm{Mnd}$:



The bialgebra law $\Delta \circ m = (m \otimes m) \circ$ (permutation) $\circ (\Delta \otimes \Delta)$ comes from the 4-cell $m \circledwedge m \in \mathrm{Mnd} \circledwedge \mathrm{Mnd}$.

(-) $\otimes$ Adjunction and Adjunction $\otimes$ (-)

Adj := free 2-category on an adjunction. There is a canonical inclusion $\mathrm{Mnd} \hookrightarrow \mathrm{Adj}$ that restricts to endomorphisms of the source of the left adjoint. There is a canonical epimorphism $\ell : \mathrm{Arr} \twoheadrightarrow \mathrm{Adj}$ that imposes that the arrow is a left adjoint.

Theorem (JF–Scheimbauer, JF–R, Masuda):

- $\mathrm{id} \otimes \ell : k\text{-Cell} \otimes \mathrm{Arr} \to k\text{-Cell} \otimes \mathrm{Adj}$ is the epimorphism that imposes that $\partial(k\text{-Cell}) \otimes \mathrm{Arr}$ factors through $\partial(k\text{-Cell}) \otimes \mathrm{Adj}$ and also that the $(k+1)$-dimensional filling is a left adjoint.
- $\ell \otimes \mathrm{id} : \mathrm{Arr} \otimes k\text{-Cell} \to \mathrm{Adj} \otimes k\text{-Cell}$ is the epimorphism that imposes that $\mathrm{Arr} \otimes \partial(k\text{-Cell}) \rightsquigarrow \mathrm{Adj} \otimes \partial(k\text{-Cell})$ and also that the mate of the $(k+1)$-dim filling is a left adjoint.

Provided f, g are left adjoints, the mate of $\phi : f \Rightarrow g$ is the canonical map $\phi^{\curvearrowright} : g^R \Rightarrow f^R$.

Adj $\owedge$ Adj $\rightsquigarrow$ Retract $\rightsquigarrow$ coherence proof

Corollary: $L_3(\mathrm{Adj} \owedge \mathrm{Adj})$ = the free 3-category on a retract $\left(1_{\mathcal{C}} \xrightarrow{f} X \xrightarrow{g} 1_{\mathcal{C}},\ \alpha : \mathrm{id}_{1_{\mathcal{C}}} \overset{\sim}{\Rrightarrow} gf\right)$ in which

- f is a left adjoint and g is a right adjoint,
- and the canonical 2-morphism $\alpha^{\curvearrowright} : f^R \Rightarrow g$ is a right adjoint, equiv. canonical 2-morphism $\alpha^{\curvearrowleft} : g^L \Rightarrow f$ is a left adjoint.

Main coherence statement: $\mathrm{Mnd} \hookrightarrow \mathrm{Adj}$ and so

$$\overbrace{L_3(\mathrm{Mnd} \owedge \mathrm{Mnd})}^{\text{walking bialgebra}} \quad \to \quad \overbrace{L_3(\mathrm{Adj} \owedge \mathrm{Adj})}^{\text{walking adjunctible retract}}$$

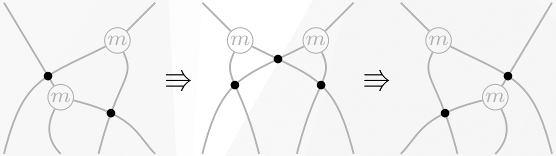
$$\underbrace{\mathrm{Fun}^{\mathrm{oplax}}(L_3(\mathrm{Mnd} \owedge \mathrm{Mnd}), \mathcal{C})}_{\mathbf{BiAlg}(\Omega^2\mathcal{C})} \quad \leftarrow \quad \underbrace{\mathrm{Fun}^{\mathrm{oplax}}(L_3(\mathrm{Adj} \owedge \mathrm{Adj}), \mathcal{C})}_{\mathbf{Ret}^{\mathrm{adj}}(\mathcal{C})}$$

The antipode axiom

A bialgebra B is Hopf if and only if the shear map

$$B \otimes B \xrightarrow{\Delta_B \otimes \mathrm{id}_B} B \otimes B \otimes B \xrightarrow{\mathrm{id}_B \otimes m_B} B \otimes B$$

is invertible. In $\mathrm{Mnd} \owedge \mathrm{Mnd}$, the universal shear map looks like:

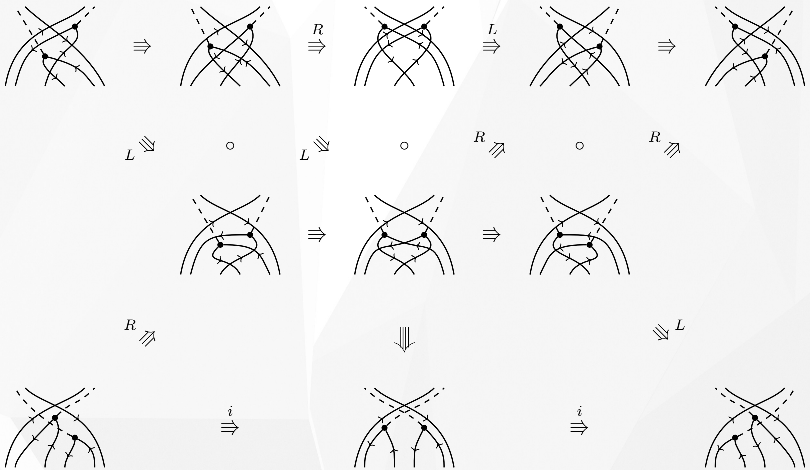


The bialgebra B is the intersection point. The light gray shows the cells in $\mathrm{Mnd} \otimes \mathrm{Mnd}$ that are smashed out in $\mathrm{Mnd} \owedge \mathrm{Mnd}$.

Theorem: The universal shear map inverts when sent via $\mathrm{Mnd} \otimes \mathrm{Mnd} \to \mathrm{Adj} \otimes \mathrm{Adj}$ and 3-localized.

Proof of existence of antipode

Theorem: The universal shear map inverts when sent via $\mathrm{Mnd} \otimes \mathrm{Mnd} \to \mathrm{Adj} \otimes \mathrm{Adj}$ and 3-localized. **Proof:**



(plus some technical manoeuvres with orientals in order to justify the use of strict graphical calculus)