ALL THE SEMISIMPLE n-CATEGORIES

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It's a pleasure to speak here. Everything I will say is joint work in progress with David Reutter. We've been talking about parts of this project for a while now, so many of you have seen other parts of the story. My hope, even for those of you who do know what we're doing, is to tell you another aspect of it.

These written notes are really for the speaker's benefit. In the talk itself, I might cover less or different material.

1. All the semisimple 1-categories

Let $\mathbb{k}$ be a field, of characteristic zero to make things easier. Let's list all the finite semisimple 1-categories over $\mathbb{k}$ — i.e. the space of objects of $\mathbf{2Vec}_{\mathbb{k}}$. I will write this as $\mathcal{V}^2 := \mathbf{Mod}^2_{\mathbb{k}}$, where I write **Mod** for the finitely generated projective modules. I'll also write $\mathcal{V} := \mathbf{Vec}_{\mathbb{k}}$, so that $\mathcal{V}^2 = \mathbf{Mod}_{\mathcal{V}}$.

Well, given $\mathcal{C} \in \mathcal{V}^2$, I can decompose it *canonically* as a direct sum $\mathcal{C} = \bigoplus \mathcal{C}_i$ where i ranges over the simple factors of the commutative algebra $\operatorname{End}(\mathrm{id}_{\mathcal{C}}) =: \Omega \operatorname{End} \mathcal{C}$. Moreover, $\mathcal{C}_i = \mathbf{Mod}_{A_i}$ for some canonical division ring A_i, and the centre of A_i is the corresponding summand of $\Omega \operatorname{End} \mathcal{C}$. Said better: $\mathcal{C}$ "spreads out" canonically over the $\mathbb{k}$-scheme $\operatorname{Spec}(\Omega \operatorname{End} \mathcal{C})$ as a gerbe. How to classify division rings? You prescribe the centre $\mathbb{L}$, which is some finite-degree field extension of $\mathbb{k}$, and then ask for a class in the Galois cohomology $\mathrm{H}^2_{\mathbb{G}_m}(\mathbb{L})$. Let's write all these data together in terms of Galois descent. Let $\bar{\mathbb{k}}$ denote the algebraic closure of $\mathbb{k}$; then $\operatorname{Spec}(\Omega \operatorname{End} \mathcal{C})(\bar{\mathbb{k}})$ is a finite set X and, by semisimplicity, completely determines $\Omega \operatorname{End} \mathcal{C} \otimes \bar{\mathbb{k}}$; the scheme $\operatorname{Spec}(\Omega \operatorname{End} \mathcal{C})$ is the same data as this set X together with an action by $\operatorname{Gal}(\bar{\mathbb{k}}/\mathbb{k})$; the gerbe is a G-equivariant twisted cohomology class on X valued in $\mathbb{G}_m(\bar{\mathbb{k}}) = \bar{\mathbb{k}}^\times$ (with its canonical G-action). All together, we find an equivalence *of spaces*:

$$\operatorname{ob} \mathcal{V}^2 = \{\text{finite set } X \text{ with } \operatorname{Gal}(\bar{\mathbb{k}}/\mathbb{k})\text{-action and } \operatorname{Gal}(\bar{\mathbb{k}}/\mathbb{k})\text{-equiv } \mathrm{H}^2_{\bar{\mathbb{k}}^\times}\text{-class}\}$$

Is this computable? It requires you to know the Galois group G and the cyclotomic character, and it requires you to be good at computing cohomology of profinite groups. Which is hard, but at least it's all "classical" algebraic topology. All the field/linear/noninvertible stuff has been gotten rid of.

What if we wanted to know all the semisimple supercategories, or more generally the categories tensored-and-enriched over some symmetric fusion category $\mathcal{E}$? This is still a "Galois" problem. It turns out that, even if $\bar{\mathbb{k}}$ is algebraically closed *as a zero-categorical ring*, $\mathbf{Vec}_{\bar{\mathbb{k}}} = \mathbf{Mod}_{\bar{\mathbb{k}}}$ is not algebraically closed. What I mean by this is the following: by the Nullstellensatz, a commutative

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ring R is an *algebraically closed field* if and only if every sufficiently-finite nonzero commutative ring map $R \to S$ splits. Well, $\mathbf{Vec}_{\bar{\Bbbk}} \to \mathbf{sVec}_{\bar{\Bbbk}}$ is definitely finite, and definitely does not split. On the other hand, Deligne's *existence of super fibre functors* says that $\mathbf{sVec}_{\bar{\Bbbk}}$ is algebraically closed, and *Tannakian duality* can be thought of as the statement of Galois descent, so that $\mathbf{Vec}_{\bar{\Bbbk}} \to \mathbf{sVec}_{\bar{\Bbbk}}$ is a Galois extension. Going higher, let's write $\mathcal{W}^n$ for the algebraic closure of $\mathcal{V}^n_{\Bbbk}$: so $\mathcal{W}^0 = \bar{\Bbbk}$ and $\mathcal{W}^1 = \mathbf{sVec}_{\bar{\Bbbk}}$. It turns out that $\mathcal{W}^2 = \mathbf{Mod}_{\mathcal{W}^1} = \{$super algebras and superbimodules over $\bar{\Bbbk}\}$. Pick up an object $\mathcal{C} \in \mathcal{W}^2$. Again it decomposes *canonically* as a sum of simples, indexed by $\operatorname{Spec}(\Omega \operatorname{End} \mathcal{C})$ — that's a priori a commutative superalgebra over $\bar{\Bbbk}$, but it's semisimple, so it's still just an algebra over $\bar{\Bbbk}$, and the spectrum is just a set — and moreover in $\mathcal{W}^2$, all simples are invertible, i.e. elements of $\mathbb{G}_m(\mathcal{W}^2)$. That's some symmetric monoidal 2-groupoid. Who is it? Well, you know that $\pi_2 \mathbb{G}_m(\mathcal{W}^2) = \pi_0 \mathbb{G}_m(\mathcal{W}^0) = \bar{\Bbbk}^\times$. You also know that $\pi_1 \mathbb{G}_m(\mathcal{W}^2)$ is the set of invertible super vector spaces, i.e. the even and odd lines, and $\pi_0 \mathbb{G}_m(\mathcal{W}^2)$ is the Morita classes of invertible superalgebras $\operatorname{Cliff}(0)$ and $\operatorname{Cliff}(1)$.

This list $\bar{\Bbbk}^\times, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ is familiar: it is the character groups of the stable homotopy groups of spheres. This is not a coincidence. Suppose that you have a symmetric monoidal n-groupoid T, and all you know about it is that $\pi_n T = \bar{\Bbbk}^\times$. What are the possible Ts that it could be? What are the universal choices? Consider the higher category of all such T's, and look at its homotopy 1-category. This homotopy 1-category has an initial object: $\mathrm{H}^n_{\bar{\Bbbk}^\times}$. That's pretty obvious: that's the groupoid $K(\bar{\Bbbk}^\times, n)$ with its canonical symmetric monoidal structure, and when I said $\pi_n T = \bar{\Bbbk}^\times$, I could have said that there was a map $K(\bar{\Bbbk}^\times, n) \to T$ (since T is an n-groupoid by assumption). Are there any other universal T's? It turns out that, using the fact that $\bar{\Bbbk}^\times$ is a divisible and hence injective abelian group, this category also has a terminal object, denoted $\mathrm{I}^n_{\bar{\Bbbk}^\times}$. It is characterized by: for any symmetric monoidal n-groupoid A, $\hom(A, \mathrm{I}^n_{\bar{\Bbbk}^\times}) = \hom(\pi_n A, \bar{\Bbbk}^\times)$, where the LHS means homs in the homotopy category of symmetric monoidal n-groupoids, and the RHS means homs of abelian groups. By testing against (truncations of suspensions of) the sphere spectrum, you find that $\pi_m \mathrm{I}^n_{\bar{\Bbbk}^\times}$ is the character group of the $(n-m)$th stable homotopy group of spheres.

In particular, if all we know about $\mathcal{W}^2$ is that $\mathcal{W}^0 = \bar{\Bbbk}$, then we already know that $\mathbb{G}_m(\mathcal{W}^2)$ admits a canonical [in the homotopy category] map to $\mathrm{I}^2_{\bar{\Bbbk}^\times}$. Then computing with superalgebras can show that this map is an isomorphism.

Ok, so what do we find?

$$\operatorname{ob} \mathcal{W}^2 = \{\text{finite set } X \text{ plus a class in } \mathrm{I}^2_{\bar{\Bbbk}^\times}(X)\}$$

This is an equivalence of spaces in the following sense: the fundamental group at (X, ω) is $\mathrm{I}^1_{\bar{\Bbbk}^\times}(X)$, etc. Suppose you care about, say, $\operatorname{ob} \mathcal{E}^2$ for some symmetric fusion 2-category $\mathcal{E}^2$. Well, $\mathcal{E}^2 \to \mathcal{W}^2$ is Galois, and

$$\operatorname{ob} \mathcal{E} = \{\text{finite set } X \text{ with a } \operatorname{Gal}(\mathcal{W}^2/\mathcal{E}^2)\text{-action plus a Gal-equivariant class in } \mathrm{I}^2_{\bar{\Bbbk}^\times}(X)\}.$$

For example, $\operatorname{Gal}(\mathcal{W}^2/\mathcal{V}^2) = \operatorname{Gal}(\mathcal{W}^1/\mathcal{V}^1)$, and if $\Bbbk = \bar{\Bbbk}$ was already algebraically closed then this is just a $B\mathbb{Z}/2\mathbb{Z}$. So it must act trivially on the set X, and the equivariance is just fixed data on the class in $\mathrm{I}^2_{\bar{\Bbbk}^\times}(X)$. But $(\mathrm{I}^2_{\bar{\Bbbk}^\times})^{B\mathbb{Z}/2\mathbb{Z}} = \mathrm{H}^2_{\bar{\Bbbk}^\times}$, and you recover the earlier description. Warning: these formulas become more complicated as you increase "2".

2. All the semisimple 3-categories

Let's temporarily skip over the 2-categories, and jump to $\operatorname{ob} \mathcal{V}^4 = \{$finite semisimple 3-categories$\}$. My strategy will be: work out the algebraic closure; conclude the answer through Galois descent. I'll write $\mathcal{W}^4$ for the algebraic closure of $\mathcal{V}^4$.

Pick up $\mathcal{C} \in \operatorname{ob} \mathcal{W}^4$ (or $\operatorname{ob} \mathcal{V}^4$ for that matter). We can decompose it into a sum of simples indexed by the set $\operatorname{Spec}(\Omega^3 \operatorname{End} \mathcal{C})$. Dimension count: $\mathcal{C}$ is a 3-category, so $\operatorname{End} \mathcal{C}$ is a monoidal 3-category, so $\Omega^3 \operatorname{End} \mathcal{C}$ is a monoidal 0-category i.e. a ring. In the 1D case, we implicitly used that when $\mathcal{C}$ is a 1-category, $\Omega \operatorname{End} \mathcal{C}$ is an E_2-algebra and hence commutative. Now we have more commutativity: the monoidal 1-category $\Omega^2 \operatorname{End} \mathcal{C}$ is actually E_3-monoidal, and so again commutative.

I already mentioned Tannakian duality: commutative monoidal 1-categories are "the same" as groupoids. This is more sharply true over $\mathcal{W}^1$ than over $\mathcal{V}^1$: for the latter, you need to remember some "emergent fermion" information, which is the Galois descent data.

So, if I pick up $\mathcal{C} \in \operatorname{ob} \mathcal{W}^4$, then I get this canonical finite 1-groupoid $X = \operatorname{Spec}(\Omega^2 \operatorname{End} \mathcal{C})$. And $\mathcal{C}$ canonically "spreads out" over X: there is a bundle over X whose global sections are $\mathcal{C}$, and the fibres $\mathcal{C}_x$ of this bundle have $\operatorname{Spec}(\Omega^2 \operatorname{End} \mathcal{C}) = \mathrm{pt}$. In the previous case, I used that, once I'm algebraically closed, a simple 1-category was invertible. The statement now is: if(f) $\mathcal{C}_x \in \operatorname{ob} \mathcal{W}^4$ has $\operatorname{Spec}(\Omega^2 \operatorname{End} \mathcal{C}_x) = \mathrm{pt}$, then $\mathcal{C}_x$ is invertible.

What do we know about the invertibles in $\mathcal{W}^4$? Well, we do know that they form a symmetric monoidal 4-groupoid $\mathbb{G}_m(\mathcal{W}^4)$, and that $\pi_4 \mathbb{G}_m(\mathcal{W}^4) = \bar{\Bbbk}^\times$, and the homotopy category of these has a terminal object $\mathrm{I}^4_{\bar{\Bbbk}^\times}$, so you do get a canonical comparison map $\mathbb{G}_m(\mathcal{W}^4) \to \mathrm{I}^4_{\bar{\Bbbk}^\times}$.

Fact: Algebraic closedness of $\mathcal{W}^n$ implies that the canonical comparison map $\mathbb{G}_m(\mathcal{W}^n) \to \mathrm{I}^n_{\bar{\Bbbk}^\times}$ is an isomorphism (on torsion subgroups[1]). In fact, this almost characterises $\mathcal{W}$:

Theorem [JF–Reutter]: $\mathcal{W}^n$ is uniquely determined by:

(1) $\mathcal{W}^\bullet$ is an inductive limit of finite semisimple extensions of $\mathcal{V}^\bullet$.
(2) The canonical (in the homotopy category) map $\mathcal{W}^1 \to \mathbf{sVec}_{\bar{\Bbbk}}$ is an isomorphism.
(3) The canonical (in the homotopy category) map $\mathbb{G}_m(\mathcal{W}^\bullet) \to \mathrm{I}^\bullet_{\bar{\Bbbk}^\times}$ is a (torsion) isomorphism.

All together, you find:

Corollary:

$$\operatorname{ob} \mathcal{W}^4 = \{\text{finite groupoid } X \text{ with class in } \mathrm{I}^4_{\bar{\Bbbk}^\times}(X)\}$$

$$\operatorname{ob} \mathcal{V}^4 = \{\text{finite groupoid } X \text{ with action by } \operatorname{Gal}(\mathcal{W}^4/\mathcal{V}^4) \text{ and Gal-equivariant class in } \mathrm{I}^4_{\bar{\Bbbk}^\times}(X)\}$$

There is still the problem of computing the Galois group. But at least it tells you that the problem is answerable in classical homotopy theory. Note that only the 1-type of $\operatorname{Gal}(\mathcal{W}^4/\mathcal{V}^4)$ can act nontrivially on X; so the 2-connective cover, since it acts trivially on X, just is reducing the cohomology problem from $\mathrm{I}^4_{\bar{\Bbbk}^\times}(X)$ to whatever the fixedpoint spectrum is.

A version of this Corollary was stated by Lan–Kong–Wen.

3. All the semisimple $(2n-1)$-categories

Exactly the same approach handles $\mathcal{W}^{2n}$. You pick up $\mathcal{C} \in \operatorname{ob} \mathcal{W}^{2n}$. You look at $\Omega^n \operatorname{End} \mathcal{C}$. This is E_{n+1}-monoidal of category number $n-1$, hence commutative, hence there is an $(n-1)$-type $X = \operatorname{Spec}(\Omega^n \operatorname{End} \mathcal{C})$. $\mathcal{C}$ spreads out as a bundle over X whose fibres $\mathcal{C}_x$ have $\operatorname{Spec}(\Omega^n \operatorname{End} \mathcal{C}_x) = \mathrm{pt}$.

Theorem [JF-R]: $\mathcal{C} \in \operatorname{ob} \mathcal{W}^{2n}$ is invertible iff $\operatorname{Spec}(\Omega^n \operatorname{End} \mathcal{C}) = \mathrm{pt}$.

Corollary:

$$\operatorname{ob} \mathcal{W}^{2n} = \{\text{finite } (n-1)\text{-groupoid } X \text{ with class in } \mathrm{I}^{2n}_{\bar{\Bbbk}^\times}(X)\}$$

and the $\mathcal{V}^{2n}$-case is accessible by Galois descent.

Here is the basic idea of the Theorem. $\operatorname{End} \mathcal{C}$, being a full "matrix" algebra, satisfies a nondegeneracy aka "trivial centre" condition. It is filtered by $\Omega^k \operatorname{End} \mathcal{C}$, and there is a sort of "associated graded" $\Omega^k \operatorname{End} \mathcal{C} / \Omega^{k+1} \operatorname{End} \mathcal{C}$, which is a finite set that we denote $\pi_k \operatorname{End} \mathcal{C}$. The theorem is that the "associated graded" sets have a pairing on them $\pi_k \operatorname{End} \mathcal{C} \times \pi_{n-1-k} \operatorname{End} \mathcal{C} \to \bar{\Bbbk}$, and the nondegeneracy condition implies (and is equivalent to) this pairing being an invertible and in particular square matrix. So this is a "Poincaré duality" statement for objects in $\mathcal{W}^{2n}$. But if $\Omega^n \operatorname{End} \mathcal{C}$ is

[1] One would prefer a statement which gave an isomorphism full stop. But it isn't. It turns out that there is some nontorsion in $\mathbb{G}_m(\mathcal{W}^n)$: ignoring the stuff in degree 0, we have $\mathbb{G}_m(\mathcal{W}^n) \otimes \mathbb{Q} = \mathrm{H}^{n-4}_{\mathbb{Q}^\infty}$. It is concentrated in just one degree, and it's there for basically the same reason that topology in dimension 4 is wild. Since all my spaces will be (π-)finite and my Galois groups will be pro(π-)finite, they will not be able to see this difference.

trivial, then half of the $\pi_k \operatorname{End} \mathcal{C}$'s are trivial, and so the other half are trivial by Poincaré duality, and so $\mathcal{C}$ has trivial endomorphisms, and so $\mathcal{C}$ is invertible.

4. What about higher gauge theories?

Look again at that Corollary. What if you started with a higher groupoid X which was not $(n-1)$-truncated? You could still pick a class $\omega \in \mathrm{I}^{2n}_{\bar{\Bbbk}^\times}(X)$ — this is some "higher gerbe" data — and look at its global sections. The Corollary says that the space you get also has a description in terms of some other $(n-1)$-truncated groupoid X'.

You know this in practice. Let's look when $n = 2$ and $X = \mathrm{B}G$ for a finite group G, and look at the case when $\omega \in \mathrm{H}^2_{\bar{\Bbbk}^\times}(X)$. Then what you end up computing is the category of ω-projective representations of G, equivalently the module category for the twisted group algebra $\bar{\Bbbk}^\omega[G]$.

It's possible for this category to be equivalent just to **Vec**. Indeed, that happens precisely when there is a unique irreducible ω-projective representation. Here's an example. Suppose that $G = A$ is an abelian group. Then it is a classical fact that $\mathrm{H}^2_{\bar{\Bbbk}^\times}(\mathrm{B}A) \cong \mathrm{Alt}^2(A^\vee)$ is the space of $\bar{\Bbbk}^\times$-valued alternating 2-forms on A — bilinear forms on A such that $\omega(a,a) = 1$ for all $a \in A$. The "unique irrep" condition is equivalent to asking ω to be *nondegenerate*: the induced map $A \to A^\vee = \hom(A, \bar{\Bbbk}^\times)$ sending $a \mapsto \omega(a,-)$ is an iso.

If I do write down a twisted group algebra, then not only do I have its potentially-trivial category of modules, but also I have its regular module. Let's suppose that A is abelian and ω is nondegenerate, and that I *choose* a trivialization of this category. (There's a $B\bar{\Bbbk}^\times$-torsor worth of choices there.) Once I've done this, then I get the trivial category **Vec** equipped with a nontrivial element. You can work out which element it is up to isomorphism: A necessarily has dimension d^2 for some $d \in \mathbb{N}$, and you get the d-dimensional vector space this way.

For every d, there is an abelian group with a nondegenerate alternating form of dimension d^2. So we accidentally produced all the (nonzero) vector spaces this way. We didn't quite get an exact list. . . .

5. Abelian Chern–Simons theories

Let's do this again but now for arbitrary n. Rather than using an $(n-1)$-groupoid with an $\mathrm{I}^{2n}_{\bar{\Bbbk}^\times}$-class, let's try a $K(A,n)$. Since $K(A,n)$ is pointed, it's natural to split any cohomology $E(K(A,n)) = E(\mathrm{pt}) \oplus \tilde{E}(K(A,n))$, where $\tilde{E}$ denotes the cohomology classes trivialized at the basepoint. It's really $\tilde{H}^2$ that classifies twisted group algebras, but you didn't notice the difference because $\mathrm{H}^2(\mathrm{pt}) = 0$.

Proposition [JF-R]: $\tilde{\mathrm{I}}^{2n}_{\bar{\Bbbk}^\times}(K(A,n)) = \mathrm{skew}^n \operatorname{Sym}^2(A^\vee)$ is the space of $(\mathrm{skew})^n$-symmetric bilinear forms on A. In other words, it is the space of symmetric or skew-symmetric forms depending on n. My bilinear forms are always valued in $\bar{\Bbbk}^\times$, and the axiom is that $\omega(b,a) = \omega(a,b)^{(-1)^n}$. **Technique:** Goodwillie calculus.

Example: We said already that $\mathrm{H}^2(K(A,1))$ is the space of alternating forms, and classifies twisted group algebras. $\tilde{\mathrm{I}}^2$ classifies twisted group superalgebras. A skew form might not be alternating: $\omega(a,a)$ just has to be in ± 1 and not necessarily 1. This number $\omega(a,a)$ records whether the a-graded piece of the group superalgebra is bosonic or fermionic. For instance, $A = \mathbb{Z}/2\mathbb{Z}$ admits a nondegenerate (!) skew form $\omega(a,b) = (-1)^{ab}$. Its twisted group algebra is $\mathrm{Cliff}(1)$, which is invertible but not Morita trivial.

Example: You surely know Eilenberg and Mac Lane's fact that $\mathrm{H}^4(K(A,2))$, which classifies braided fusion categories with fusion rules A, is precisely the space of quadratic forms on A. I'm telling you that $\tilde{\mathrm{I}}^4(K(A,2))$ is the space of symmetric bilinear forms. The map $\mathrm{H}^4(K(A,2)) \to \tilde{\mathrm{I}}^4(K(A,2))$ is surjective but not injective. For example, it identifies $\mathbf{Vec}[\mathbb{Z}/2\mathbb{Z}]$ and **sVec**; it identifies the semion with the antisemion. The symmetric bilinear form that you remember is precisely the S-matrix; what you forget is the T-matrix.

Remark: I've stated the Proposition at the level of sets — I've told you π_0 of the space of $\tilde{\mathrm{I}}^{2n}$-classes. At the level of spaces, there is only one other homotopy group: $\pi_n = A^\vee$.

Theorem: Pick $\omega \in \tilde{\mathrm{I}}^{2n}_{\mathbb{k}^\times}(K(A,n))$. The space $\int_{K(A,n)} \omega$ of global sections of this gerbe is invertible (in $\mathcal{W}^{2n}$) iff ω is *nondegenerate* in the sense that the induced map $A \mapsto A^\vee$ is an iso.

But the invertibles in $\mathcal{W}^{2n}$ were I^{2n}. So what we have is a fancy map

$$\{\text{finite abelian groups with nondegenerate (skew)symmetric bilinear form}\} \to \mathrm{I}^{2n}.$$

This map turns out to be a sort of "Gauss sum" map, and the invertibility turns out to boil down to the invertibility of Gauss sums.

6. All the semisimple $(2n-2)$-categories

Actually, it turns out that $\int_{K(A,n)} \omega$ is almost always not only invertible but actually trivializable. (Proof: Compute all answers, and cite hard results about I^{2n}.) There's a difference, of course, between "trivializable" and "trivialized." Anyway, pick a trivialization of $\int_{K(A,n)} \omega$.

On the other hand, because ω started out as trivialized at the basepoint in $K(A,n)$, there is a section of this bundle — it is a "delta function" or "skyscraper sheaf" or whatever you want to call it at the basepoint. So that's an element of $\int_{K(A,n)} \omega \in \mathcal{W}^{2n}$, which I've trivialized. So that's an element of $\mathcal{W}^{2n-1}$.

All together, we get a map

$$\left\{ \begin{array}{l} \text{finite abelian group } A \text{ with a (nondegenerate) class } \omega \in \tilde{\mathrm{I}}^{2n}(K(A,n)) \\ \qquad\qquad\qquad\qquad \text{and a trivialization of } \displaystyle\int_{K(A,n)} \omega \in \mathrm{I}^{2n} \end{array} \right\} \to \mathrm{ob}\,\mathcal{W}^{2n-1}.$$

Let me give a name to the LHS. I'll call it $\mathfrak{gauss}^{2n}$. I invite suggestions for better names.

Remark: If you replace "$2n$" with an odd number m, then you'd be asking about "$K(A, \frac{odd}{2})$", and there aren't any of those. Well, there's one of them: pt counts. The trivialization of $\int_{\mathrm{pt}} 1 \in$ is a choice of element in I^{m-1}. So really you have this fibre sequence

$$\mathrm{I}^{\bullet-1} \to \mathfrak{gauss}^\bullet \to \{K(A, \bullet/2) \text{ with a nondegenerate class } \omega \in \tilde{\mathrm{I}}^\bullet(K(A,\bullet/2))\}.$$

Finally, pick up a generic $\mathcal{C} \in \mathrm{ob}\,\mathcal{W}^{2n-1}$. Look at $\Omega^n \operatorname{End} \mathcal{C}$. This is an E_{n+1}-monoidal $(n-2)$-category, and hence commutative, and hence its spectrum is an $(n-2)$-type, and $\mathcal{C}$ spreads out over this spectrum. The fibre $\mathcal{C}_x$ at $x \in X = \operatorname{Spec}(\Omega^n \operatorname{End}\mathcal{C})$ has $\operatorname{Spec}(\Omega^n \operatorname{End}\mathcal{C}_x) = \mathrm{pt}$.

Theorem [JF-R]: Suppose that $n \geq 2$. Then, $\operatorname{Spec}(\Omega^n \operatorname{End}\mathcal{C}) = \mathrm{pt}$ iff $\mathcal{C}$ is in $\mathfrak{gauss}^{2n}$.

Corollary: If $n \geq 2$,

$$\mathrm{ob}\,\mathcal{W}^{2n-1} = \{(n-2)\text{-type } X \text{ with a map } X \to \mathfrak{gauss}^{2n}\}.$$

7. What about the Galois group?

The same techniques tell us a lot about $\operatorname{Gal}(\mathcal{W}/\mathcal{V})$. I'll just report the answer. There was this "cyclotomic character" $\mathrm{B}\operatorname{Gal}(\mathcal{W}^\bullet/\mathcal{V}^\bullet) \to \mathrm{B}\operatorname{Aut}(\mathrm{I}^\bullet)$. The fibre of this map has a map to (the dual to) the "L-theory" whose cocycles are $\mathfrak{gauss}$. This map is almost an isomorphism: it fails only in dimension ≈ 4. In principle, this should supply a complete description of $\mathrm{ob}\,\mathcal{V}^n$, subject to your ability to run spectral sequences.

8. What fails in low dimension?

If you want to understand $\mathcal{W}^1$ or $\mathcal{W}^3$, this method fails basically because matrix algebras and nondegenerate braided fusion categories don't come with canonical gradings by abelian groups. For example, the simple objects in $\mathcal{V}^3$ are exactly classified by Morita equivalence classes of fusion categories; recall that two fusion categories are Morita equivalent iff their Drinfeld centres are isomorphic, so that the set of simples in $\mathcal{V}^3$ is the set of Witt-trivializable braided fusion categories. What my method gives you is are just (those fusion categories whose Drinfeld centres are) the "pointed" (grouplike fusion) categories.

The magic in higher dimensions is that the "pointedness" / grouplike fusion is automatic. At the end of the day, it's basically the same magic as the classical fact that higher homotopy groups are abelian, although the relation is a bit roundabout. The $\mathcal{W}^1$ and $\mathcal{W}^3$ cases are the nonabelian π_0 and π_1, and the doubling is the Poincaré duality.