The Universal Target Category

JHU Topology Seminar

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Based on jt work in progress w/ David Reutter

And inspiration from Freed-Scheimbauer-Teleman, Schenk et al, and Hopkins.

① Motivation

Suppose you have a quantum system w/ some species of particle-like excitations. They can move around, and convert into each other: they form a category. This category has:

⊕ superposition

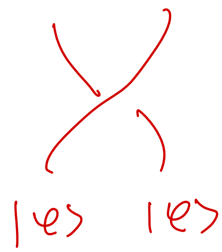
⊗ fusion

In $\geq 3+1$ D, it is sym monoidal.

Question: Can you consistently assign a v-space of "internal states" to every particle?

i.e. $\exists$? sym $\otimes$ $\mathcal{C} \to \mathrm{Vec}$?

<u>Answer</u>: Not necessarily. E.g. take two protons, prepare them in identical states.

Then  should give $|\psi\rangle \otimes |\psi\rangle$ back, but in fact gives $-|\psi\rangle \otimes |\psi\rangle$.

<u>Cheap fix [Dirac]</u>: work w/ sVec: $\mathbb{Z}/2$-graded Vec w/ modified $\times$.

<u>Theorem [Deligne]</u>: This fix suffices. If $\mathcal{A}_{\neq 0}$ is sym $\otimes$ $\mathbb{C}$-linear and not too large, then $\exists\ \mathcal{A} \to \mathrm{sVec}$.

<u>Compare</u>: Maybe you just want to represent local operators. Maybe just those that move top'lly. These form an $\mathbb{R}$-alg, comm in $\geq 1+1$ D.

<u>Question</u>: $\exists$? $A \to \mathbb{R}$? <u>Answer</u>: NO.

<u>Cheap fix [Cardano]</u>: Work w/

$\mathcal{C} = \{\mathbb{Z}/2\text{-graded } \mathbb{R}\text{-numbers w/ modified multiplication}\}$

<u>Theorem [Hilbert]</u>: This suffices.

Can also go higher. A typical quantum system can have "extended objects".
E.g. defects in a lattice where the crystal structure doesn't align.
These form higher categories.

<u>General Question</u>: Find a universal target higher category where any not-too-large sym $\otimes$ cat maps to.

② Existence + Characterization

To organize all levels at the same time, I'll use something this dept has some expertise in: categorical spectra. A cat sp_ is a sequence $\mathcal{C}^0, \mathcal{C}^1, \ldots$ of pointed higher categories w/

$$\Omega\, \mathcal{C}^n := \mathrm{End}_{\mathcal{C}^n}(\mathrm{pt}_{\mathcal{C}^n}) \simeq \mathcal{C}^{n-1}.$$

I will use cat sp w/:

* $\mathcal{C}^n$ is a weak (aka (∞,n)) - cat.
* $\mathcal{C}^0 \simeq \mathbb{Q}$.
* $\mathcal{C}^n$ is Cauchy complete.
* $\mathcal{C}^n$ has all adjoints.

I tend to say tower for such a cat sp. But I'm open to vocab suggestions.

Example: $\mathbb{Q}$, $\mathrm{Mod}^{f.d.}(\mathbb{Q})$, $\mathrm{Mod}^{f.d.}(\mathrm{Mod}^{f.d.}(\mathbb{Q}))$, ...

$\equiv$ "$\mathbb{Q}^{\infty}$"!

Defn: A tower A° is <u>separably closed</u> (aka <u>sep'ly Nullstellensatzian</u>) if $\forall$ $A^\circ \to B^\circ$ w/ $B \neq 0$ and B a sufficiently finite extension, then $\exists$ a splitting $A^\circ \leftarrow B^\circ$.

Goal: Build + study the separable closure W° of $\mathbb{Q}^\circ$ (or $\mathbb{R}^\circ$ or ...)

Some features of W° (if it exists):

(i) $W^0 \cong \overline{\mathbb{Q}}$ (or $\mathbb{C}$ or ...)

$W^1 \cong \mathrm{sVec}_{\overline{\mathbb{Q}}}$. [Deligne]

If $\mathcal{C}^\circ$ any tower, then $\exists$ a spectrum (in the correct sense) $\mathbb{G}_m(\mathcal{C}^\circ) =$ invertible objs, mors, in $\mathcal{C}$.

And $\mu(\mathcal{C}^\circ) := \mathbb{G}_m(\mathcal{C}^\circ)[\mathrm{tor}] =$ the ind-π-finite approximation of $\mathbb{G}_m(\mathcal{C}^\circ)$.

(ii) $\mu(\mathcal{W}^\circ) \cong I\mathbb{Q}/\mathbb{Z}$.

Pf: Test against gp alg's of π-finite spectra.

Theorem (JF-Reutter): Any $\mathcal{W}^\circ$ satisfying (i), (ii) is sep'ly closed.

What's really going on: there is a "galois space" $\pi^{et}_{\leq\infty}(\mathbb{Q}^\circ) = BGal(\mathcal{W}^\circ/\mathbb{Q})$

and (i) $\Rightarrow$ $\pi_{\leq 1} = *$

(ii) $\Rightarrow$ $H_\circ = *$.

so Hurewicz $\Rightarrow$ this space $= *$.

What we show: if $\mathcal{W}^\circ$ satisfies (i) and $\mathcal{W}^\circ \to \mathcal{B}^\circ$ is an iso on $\mathcal{W}' \to \mathcal{B}'$, then $\mathcal{W}^\circ \to \mathcal{B}^\circ$ is a solvable extension.

and:

<u>Categorical</u> <u>Kronecker-Weber Thm</u>:

For a tower, (ii) $\Leftrightarrow$ $\mathcal{W}$ is <u>abelian-closed</u> (equiv: solvably closed), i.e. no nontrivial abelian extensions.

<u>Main pf ingredient</u>: Categorical Kummer thy.

Moreover, this tells you how to build $\mathcal{W}$:
Start w/ $\overline{\mathbb{Q}}$ (or $\overline{\mathbb{R}}$). Now iteratively do a bunch of Kummer extensions.

Kummer thy only classifies A-extensions if you have "Ath roots of unity"!

<u>Thm</u>: Iterative Kummer extensions can create all roots of unity.

③ Calculating the Galois SG:

Set $Gal := Gal(\mathcal{W}^\circ/\mathbb{Q}^\circ)$.

The cyclotomic character is

profinite sphere

$$Gal \longrightarrow Aut(I\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{S}}^\times \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{"} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad GL_1(\hat{\mathbb{S}})$$

action on $\mu(\mathcal{W}^\circ)$.

The fibre of this map is $\dfrac{GL_1(\hat{\mathbb{S}})}{Gal} =$

$\{(\mathcal{W}, \tau)$: $\mathcal{W}^\circ$ is an alg closure of $\mathbb{Q}^\circ$,

$\tau : \mu(\mathcal{W}^\circ) \simeq I\mathbb{Q}/\mathbb{Z}$ is a choice of equiv $\}$.

Since $\mathcal{W}^\circ$ had all duals, I can think of its elts as TQFTs.

How to construct a TQFT? One way: finite path $\int$ (aka higher semiadditivity).

Best case is <u>abelian CS thy</u>:

Given a π-finite spectrum A and an inhomogeneous quadratic function $q : A \to I\mathbb{Q}/\mathbb{Z}[n]$, look at the TQFT

$$M^n \longmapsto \int_{a : M \to A} \exp\Big(2\pi i \underbrace{\int_M q(a)}_{\in I\mathbb{Q}/\mathbb{Z}}\Big)$$

This is an "a da" type action.

<u>Thm (Gauss)</u>: This TQFT is invertible iff q is nondegenerate.

Set $\mathcal{C}$ = L-thy of π-finite spectra w/ inhom $I\mathbb{Q}/\mathbb{Z}$-valued quad functions.

$$\mathcal{C}^\vee = [\mathcal{C}, I\mathbb{Q}/\mathbb{Z}].$$

There is a map $\dfrac{\hat{\$}^{\times}}{\mathrm{Gal}} \to \ell^{\vee}$

equiv

$$\ell \times \frac{\hat{\$}^{\times}}{\mathrm{Gal}} \longrightarrow \mathrm{I}\,\mathbb{Q}/\mathbb{Z} \qquad \text{lin in } \ell$$

given by

$$((A, f), (W, \tau)) \longmapsto$$

$$\tau^{-1}\left(\text{Gauss-sum TQFT built from } (A, \tau \circ f) \right).$$

$\boxed{\underline{\text{Main Thm:}}}$ This gauss map $\dfrac{\hat{\$}^{\times}}{\mathrm{Gal}} \to \ell^{\vee}$

is almost an iso. It fails in deg 0, and the only other failure is: the con. comp of the fibre is the profinite completion of a

$$K\left(\mathbb{Z}/2^{\times\infty} \times \mathbb{Q}/\mathbb{Z}^{\times\infty}, 4\right).$$

The technique in the proof is a version of surgery theory for TQFTs instead of manifolds. Just like for manifold surgery, the one time it fails is due to "nonabelian knotting" of 1-manifolds in 3-manifolds.

The $(\mathbb{R}/2)^{\times\infty} \times (\mathbb{Q}/\mathbb{Z})^{\times\infty}$ is dual to the truly quantum Witt gp

$$\left(\frac{\text{MTCs}}{\text{morita equiv}}\right) \Big/ \left(\begin{array}{c}\text{MTCs built from} \\ \text{abelian gps w/ q. forms,} \\ \text{and Ising MTCs}\end{array}\right)$$

④ A corollary and a conjecture.

Usual surgery for framed manifolds:

$$\begin{tikzcd} PL \arrow[r, "j\text{-hom}"] \arrow[d, dashed, "?"] & GL_1(\mathbb{S}) \arrow[r] \arrow[d] & L \arrow[d, phantom, "\text{very similar, must end with.}"] \\ Gal \arrow[r, "\text{cyc character}"'] & GL_1(\hat{\mathbb{S}}) \arrow[r] & \ell^\vee \end{tikzcd}$$

surgery for TQFTs:

Conjecture: The j-homomorphism factors thru the cyclotomic character.

Enforced gaplessness thm:

Physics fact: A QFT can have a grav. anomaly living in $I\mathbb{C}^\times$. From the LES, we find:

Thm: If the anomaly of the gft is nontrivial in $I\mathbb{C}^\times$, then the gft is gapless, except for when the anomaly is an Arf-Kervaire invariant and there exist Kervaire-inv-1 framed manifolds (which happens only in dimensions 2, 6, 14, 30, 62, and possibly 126 by a famous thm of Hill-Hopkins-Ravenel).

FIN.