Rigid Firm Categories

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Quillen [1] calls a nonunital algebra A *firm* when the multiplication map $A \otimes_A A \to A$ is an isomorphism; an A-module M is firm when $A \otimes_A M \to M$ is an iso. Every unital algebra is firm, and a module for a unital algebra is unital (in the sense that the unit acts as a unit) if and only if it is firm. The many-object generalization of firm algebras to (enriched, ∞-)categories: a nonunital category C is *firm* when, for every x, z, the composition map $\mathrm{coend}_{y \in C} C(x, y) \times C(y, z) \to C(x, z)$ is an isomorphism; a presheaf f on C is firm when $\mathrm{coend}_{y \in C} C(x, y) \times f(y) \to f(x)$ is an isomorphism; a functor $f : C \to D$ of firm categories is a *firm functor* when $D(f(-), f(-))$ is firm in each variable. These notions they have been independently discovered multiple times. For example, [2] uses the term "H-unital algebra" for a firm algebra in Sp, and [3] uses the term "taxonomy" for a firm category enriched in Set.

Given a firm category C, enriched in some ambient presentably symmetric category $\mathcal{V}$, write $P_{\mathcal{V}}(C)$ for its presentable $\mathcal{V}$-category of firm $\mathcal{V}$-enriched presheaves. These categories have an Eilenberg–Watts theorem: colimit-preserving $\mathcal{V}$-functors between $P_{\mathcal{V}}(C)$ and $P_{\mathcal{V}}(D)$ are precisely the firm C-D-profunctors. One reason to care about firm categories is the following amazing result of Ramzi, which says that "firm Morita theory" fully captures the theory of dualizable presentable $\mathcal{V}$-categories. (Ramzi states the theorem differently, but this is essentially Theorem B of [4].)

Theorem ([4])**.** *For any firm $\mathcal{V}$-category C, $P_{\mathcal{V}}(C)$ is 1-dualizable in the 2-category $\mathsf{Pr}_{\mathcal{V}}$ of presentable $\mathcal{V}$-categories. Moreover, every 1-dualizable presentable $\mathcal{V}$-category is $P_{\mathcal{V}}(C)$ for some firm category C. Moreover, given a 1-dualizable presentable $\mathcal{V}$-category $\mathcal{M}$, a firm category C presenting it can be constructed as follows: choose any subspace $X \subset \mathrm{ob}(\mathcal{M})$ which is "spanning" in the sense that its full image is dense; take just the "atomic" morphisms between objects of X; then X withjust the atomic morphisms is a firm category presenting $\mathcal{M}$.*

Note that by taking a large enough object, in fact every dualizable presentable $\mathcal{V}$-category is the firm modules for a firm $\mathcal{V}$-algebra. The precise definition of "atomic morphism" is in [4]. Essentially, the atomic morphisms with domain $x \in \mathcal{M}$ are what would be the answer if x were an atomic object. Here is a valuable false etymology of the word "atomic": in Ban, the category of Banach spaces and short maps, the atomic morphisms are precisely the nuclear kernels.

One can easily talk about monoidal firm categories (and even firm-monoidal firm categories, where both the categorical composition and the monoidal product are firm rather than unital). Without units morphisms, it becomes more interesting to say the word "rigid": one cannot ask for a zig-zag composite to be an identity. Suppose that C is monoidal firm, and consider the associativity square at the level of dualizable presentable categories:

$$\begin{array}{ccc} P(C) \boxtimes P(C) \boxtimes P(C) & \xrightarrow{P(C)\boxtimes m} & P(C) \boxtimes P(C) \\ \Big\downarrow {\scriptstyle m\boxtimes P(C)} & \cong & \Big\downarrow {\scriptstyle m} \\ P(C) \boxtimes P(C) & \xrightarrow{\quad m \quad} & P(C) \end{array}$$

The sides of this square are all internal left adjoints in $\mathsf{Pr}_{\mathcal{V}}$. C is called *Gaitsgory rigid* if this square is adjointable aka Beck–Chevalley in both directions, i.e. if both mates of this square strongly commute. It is called *rigid* if it is Gaitsgory rigid and moreover the induced firm profunctors $(-)^*, {}^*(-) : C \nrightarrow C^{\mathrm{op}}$ are realized by functors.

One can moreover talk about dagger structures on firm categories. A *unitary dual functor* on a rigid monoidal dagger (firm) category is a choice of compatibility between $(-)^*$ and $(-)^\dagger$. A UDF selects a cyclic trace $\mathrm{Tr}_c : \mathrm{End}(c) \to \Omega C := \mathrm{End}(1_C)$.

The most important example of a rigid firm monoidal dagger category with UDF is a geometric bordism category ($\mathsf{Bord}^{\mathrm{Riem}}_{n,n-1}$ and its cousins). The firm representations of $\mathsf{Bord}^{\mathrm{Riem}}_{n,n-1}$ are the ones that Kontsevich and Segal [5] call *continuous*.

Let B be any rigid firm symmetric monoidal dagger category with UDF, and let $\mathbb{C}B$ denote its $\mathbb{C}$-linearization. Write $\Omega\mathbb{C}B = \mathbb{C}\Omega B$ for the $*$-algebra of endomorphisms of the unit object. The *positive cone* in $\Omega\mathbb{C}B$ is the set of all elements of the form $\mathrm{Tr}_b(f^\dagger f)$ for any $f : b \to c$ in $\mathbb{C}B$.

Definition. *A* reflection-positive partition function on B *is a positive tracial state on $\mathbb{C}B$: a $*$-monoid homomorphism $z : \Omega B \to \mathbb{C}$ whose free linearization sends positives in $\Omega\mathbb{C}B$ to positives.*

Any positive tracial state begs you to run a GNS construction. The many-object version builds, for each object $b \in B$, a Ban-valued (in fact, Hilb-valued, but I want to keep with presentable categories) firm presheaf $H_{z,b}$ on C; the traciality says that $H_{z,b}$ depends functorially on b, so that we have built a functor $H_z : B \to P_{\mathsf{Ban}}(B)$. In fact, it is better than landing just in $P_{\mathsf{Ban}}(B)$: it lands among the atomic morphisms. Let $\tilde{B}_z$ denote the "full atomic image" of $H_z : B \to P_{\mathsf{Ban}}(B)$. Then $\tilde{B}_z$ is again firm, now Ban-enriched, and again rigid symmetric monoidal dagger with UDF.

When is a firm category C Morita-equivalent to a unital one? In other words, when is $P(C)$ atomically generated? The atomic objects in $P(C)$ are the *Karoubi completion* $\mathrm{Kar}(C)$ of C: the (unital!) category whose objects the idempotents in C, and with $\mathrm{Kar}(C)(p,q) = pCq$. In the unital world, C and $\mathrm{Kar}(C)$ are always Morita equivalent; in the firm world, this happens if and only if $P(C)$ is atomically generated. Consider the case when C is Vec-enriched and has one object, i.e. when C is a firm algebra A. Build the ideal $K(A) \subset A$ generated by generated by all idempotent elements. As an idempotent ideal in a firm algebra, $K(A)$ is again firm, and by construction $K(A)$ is Morita equivalent (as a firm category) to $\mathrm{Kar}(A)$. In the unital world, $K(A) = A$, whereas in the firm world, $K(A)$ can be anywhere between A and 0. Thus one sees that A is "Morita unital" if it "has enough idempotents" — if $K(A) = A$.

Colafranceschi, Dong, Marolf, and Wang have studied in [6] precisely this question for the firm Ban-category $\tilde{B}_z$. Their main theorem is not stated this way (indeed, their paper is written in the language of quantum gravity, not mathematics, and so they do not even distinguish a "main theorem," let alone talk in term of firm Ban-categories), but their argument applies at the level of generality herein:

Theorem ([6])**.** *$\tilde{B}_z$ has enough idempotents: it is Morita equivalent to its Karoubi completion. Moreover, all endomorphism algebras in $\mathrm{Kar}(\tilde{B}_z)$ are finite-dimensional W^*-algebras.*

But $\tilde{B}_z$ is Gaitsgory-rigid and hence $\mathrm{Kar}(\tilde{B}_z)$ is as well (since Gaitsgory-rigidity is a Morita-invariant notion) and hence $\mathrm{Kar}(\tilde{B}_z)$ is rigid (since it is Karoubi complete). By construction, $\Omega\tilde{B}_z = \mathbb{C}$ (and $\Omega B \to \Omega\tilde{B}_z$ is z). The famous reconstruction result of Doplicher and Roberts [7] thus says that there is a unique (up to non-unique isomorphism) super fibre functor $\mathrm{Kar}(\tilde{B}_z) \to \mathsf{sHilb}^{\mathrm{fd}}$. Recompile via the equivalences $P_{\mathsf{Ban}}(\mathrm{Kar}(\tilde{B}_z)) \simeq P_{\mathsf{Ban}}(\tilde{B}_z)$ and $P_{\mathsf{Ban}}(\mathsf{sHilb}^{\mathrm{fd}}) = \mathsf{sBan}$, and note that the presentably-dagger-with-UDF structure on sBan is dagger-with-UDF-in-the-small-category-sense precisely on $\mathsf{sHilb} \subset \mathsf{sBan}$. One ends up with the following amazing result reconstruction result of McNamara and Wang, which will appear in upcoming work and which is what inspired this project:

Theorem (McNamara and Wang). *Let B be any rigid symmetric monoidal firm category with dagger and UDF. Let $z : \Omega B \to \mathbb{C}$ be any reflection positive partition function on B. Then z extends, uniquely up to nonunique isomorphism, to a symmetric monoidal dagger-with-UDF functor $Z : B \to \mathsf{sHilb}$.*

In other words:

Unitary functorial field theories, including the geometric ones, are determined by their partition functions.

References

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