Operators and (higher) categories in quantum field theory I

Theo Johnson-Freyd. KIAS, 19 July 2021.

Or: "Towards a definition of quantum field theory"

Roughly: QFT is to PDE as
QM is to ODE.

An ODE is a PDE in 1D.
QM $\equiv$ QFT in 1D. ← spacetime dim
↑ particle worldline.

Example of QM model is
a particle moving in some target space X. "σ-model".
ODE for functions $x : \mathbb{R}_t \to X$ ← some manifold with
s.t. $\ddot{x} = \nabla(V(x))$, $V \in \mathcal{C}^\infty(X)$. some dimension.

<u>Schrödinger</u>: a way to model QM systems is to give a Hilbert space $\mathcal{H}$ and an operator $\underbrace{\hat{H}}$.

analytic properties TBD.

In σ-model example, $\mathcal{H} = L^2(X)$, $\hat{H} = \Delta$.

The "ODE" is $\partial_t |\varphi\rangle = i\hat{H}\,|\varphi\rangle$

i.e. $U_t\,|\varphi\rangle = e^{i\hat{H}t}\,|\varphi\rangle$.

evolve by time t

<u>Problem</u>: As a definition, $(\mathcal{H}, \hat{H})$ is off by a phase.

<u>Reason</u>: $|\varphi\rangle \in \mathcal{H}$ are not physical.

Some things that are physical:

* – assoc. alg w/ some properties.

- $\mathcal{A} = B(\mathcal{H})$ alg of bounded operators.

$\mathcal{A}$ $\rightsquigarrow$ <u>states</u> linear map $\mathcal{A} \to \mathbb{C}$
$\cup\!\!\!\mid$ s.t. axioms.
a

$\cup\!\!\!\mid$
pure states

when $\mathcal{A} = B(\mathcal{H})$

pure states $= \left\{ \dfrac{\langle \psi | a | \psi \rangle}{\langle \psi | \psi \rangle} \right\}$
$ \overset{\shortparallel}{\mathbb{P}\mathcal{H}}$.

Data of $\mathcal{A}$ $\Longleftrightarrow$ data of $\mathbb{P}\mathcal{H}$.

Also physical: $\partial_t a = i[\hat{H}, a]$ $\qquad a \mapsto U_t a U_t^{-1}$.

Even $\hat{H}$ itself is not quite physical. If $\hat{H} \longmapsto \hat{H} + \underset{\mathbb{C}}{\underset{\cap}{E}}$
you haven't changed the physics.

- families of QM models $\neq$ families of $(\mathcal{H}, \hat{H})$.
  $\equiv$ projective bundles
- sym of QM models $\neq$ sym of $(\mathcal{H}, \hat{H})$
  $\subseteq PU(\mathcal{H})$ $\qquad\qquad\qquad \subseteq U(\mathcal{H})$.

- Given a projective bundle, its obstruction for being $\mathbb{P}$(vector bundle) $\in$ Brauer(parameter space) $\equiv H^2$(parameter space, $U(1)$)

$$\downarrow$$

$H^3(\text{par}, \mathbb{Z})$

- Given a projective rep

$$\begin{tikzcd} & U(\mathcal{H}) \arrow[d] \\ G \arrow[r] \arrow[ur, dotted] & PU(\mathcal{H}) \end{tikzcd}$$

$\exists$ obstruction

$$\in H^2_{gp}(G; U(1)) \longrightarrow H^3(BG; \mathbb{Z})$$

these classes are called "anomalies"

anomalies can be good: provide info about universality classes of models.

i.e. can you meaningfully write down a single QM model w/ an X-valued quantum particle?

anomaly obstructs this.

$\mathbb{P}\mathcal{H} \to$ bundle

$$\downarrow$$

$$x \in X$$

Ask: "Can you dynamicalize x"?

"$\int$ (family)"?

paths in X

Punchline 0: We should define QM / QFT "operationally" in terms of A ($\text{"}= B(\mathcal{H})$ locally")

When von Neumann tried to do this he wrote down analytic axioms "vN algebra".
He hoped to find some physically natural axiom which selected the vN algs w/ a unique simple rep.

Attempt: Demand that $Z(A) = \mathbb{C}$. "Heisenberg
$\uparrow$ centre. uncertainty principle"

!! not strong enough. vN alg A s.t. $Z(A) = \mathbb{C}$ is a vN factor. $\exists$ LoTs. Most are $\gg B(\mathcal{H})$.
Too big to admit a good state.

Stronger axiom: "Strong Noether theorem"

The monoidal category $(\mathrm{Bim}(A), \otimes_A)$ demand is equivalent to Vect (or Hilb).

$\Leftrightarrow$ A is Morita invertible.

$$\Rightarrow \; Z(A) = \mathbb{C}$$

$$\begin{array}{c} \shortparallel \\ \mathrm{End}_{\mathrm{Bin}(A)}({}_A A_A) \end{array}$$

identity object in $\mathrm{Bin}(A)$

Remark for Experts:

strong axiom $\equiv$ $Z(A) = \mathbb{C}$ and A is 2-dualizable.

If so, $A' = A^{op}$.

Then $\mathrm{Bin}(A) = \mathrm{Mod}(A \otimes A^{op})$

Defn: A vN factor is Type I when it's Morita invertible.

Given A, can ask $\exists A'$ s.t. $A \otimes A' = B(\mathcal{H})$?

$\Rightarrow$ Noether theorem: Every automorphism is innerizable. i.e.

$$Z(A)^\times \to A^\times \longrightarrow \mathrm{Aut}(A)$$
$$a \longmapsto (b \mapsto aba^{-1})$$

$$Z(A) \to A \longrightarrow \mathrm{Der}(A)$$
$$a \longmapsto [a,-]$$

are surjections

Pf: Given $f: A \xrightarrow{\sim} A$, construct ${}_A A_{fA} =: A_f$

$$\underset{A}{a} \rhd \underset{A_f}{m} \lhd \underset{A}{b} = a \cdot m \cdot f(b).$$

choose: $\exists \ {}_A A_A \cong A_f$, $1 \mapsto \hat{f}$.

Check $f(b) = \hat{f} \cdot b \cdot \hat{f}^{-1}$.

Classification thm: Indeed, all type I factors are $\cong$ $B(\mathcal{H})$ for some $\mathcal{H}$.

noncanonically

$\sum_0$ not true in families.

Family of type I factors $\neq B(\mathcal{H})$

$\equiv$ family of vN algs

Morita-invertible in world of families of algs.

not true for vN superalgebras.

$\mathcal{A}$ is the Kinematics.

Dynamics $= \hat{H}$.

I promised a niceness axiom.

In σ-model case,

$\mathcal{H} = L^2(X)$

$\mathcal{A} = B(\mathcal{H})$

$\hat{H} = \Delta + V$

<u>Fact:</u> X is compact

$\Longrightarrow$ $\exp(-t\hat{H})$ is trace class $\forall\ t > 0$.

"wick-rotated time evolution op."

$\Longleftrightarrow$ when $V = 0$.

<u>Defn:</u> A QM model is compact when $\exp(-t\hat{H})$ is trace class $\forall\ t > 0$.

E.g: Harmonic oscillator.

$\{$ compact QM models $\}$ = gapped top $\hookrightarrow$ B PU(n)

bounded ops.

$\downarrow$

$\{$ Type $\overset{\text{ordin}}{}$ I factors. $\}$ $= \mathbb{B}(\mathrm{Aut}(\mathcal{B}(\mathcal{H})))$

classifying space

is a homotopy equiv in some topologies not others. Fun.

$= \mathbb{B}\ PU(\infty)$

$= K(\mathbb{Z}, 3)$ in strong op top.

E.g.: Yes: in strong conv. of resolvent $\leftarrow$ spectrum can enter/leave ground state.

No: in "gapped" topology.

$\exists$ multiple interesting topologies.

A QM is <u>topological</u> when $\hat{H} = 0$.

<u>topological</u>* when $\hat{H} \in Z(A)$.

Compact topological $\Rightarrow$ $A = Mat(n)$.

Dream definition of QFT in higher dimensions:

- "alg" A of "operators".
  - $\hookrightarrow$ point operators] $\leftarrow$ alg str is that of Factorization alg of Costello–Gwilliam.
  - $\hookrightarrow$ extended operators.
- Noether thm: A should be "Type I"

$\Leftrightarrow$ dualizability + "$Z(A) = $ triv".

• distinguished operator playing role of $\hat{H}$.

compactness axiom.

In 1D

$\mathrm{Tr}\, \exp(-t\hat{H})$

= partition fn of S^1.

Compactness in general:

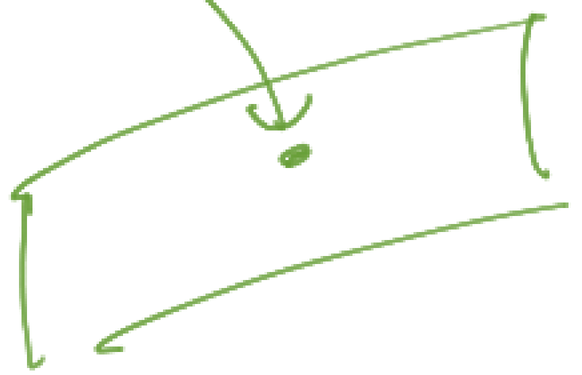
closed n-manifolds

n = spacetime dim

have finite partition function.

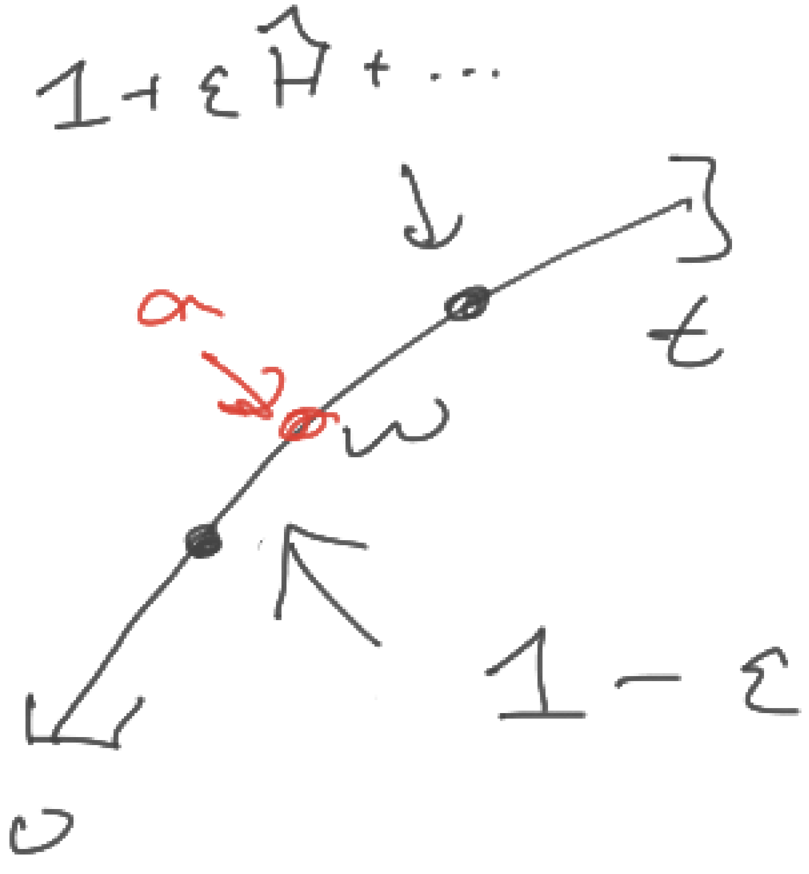
stress-energy tensor. $T_{\mu\nu}$

$T_{\mu\nu}(x)$



$= -\dfrac{\partial}{\partial g^{\mu\nu}(x)}\Bigg($  $\Bigg)$

change the metric.



Wick-rotated world line.

$$1 - \varepsilon \hat{H} + \dots = \exp(-\varepsilon \hat{H}).$$

" changing $[0, t]$ to $[0, t+\varepsilon]$.

$$\varepsilon [\hat{H}, a] = \int_{z \in S^0} \hat{H}(z) \cdot a(w)$$

nD

S^{n-1}

$$\eta^{\mu_0 \cdots \mu_{n-1}} \int_{z \in S^2} a(\omega) \quad T_{\mu_0 \nu}(z) \quad d^{n-1}_{\mu_1 \cdots \mu_{n-1}} z$$

$$= P_\nu[a](\omega).$$

$T_{\mu\nu}$ $\supseteq$

- energy-momentum.
- $so(n)$-action
- rescaling metric.

Type I $\equiv$ $\underbrace{Z(A)}_{\uparrow}$ + $\underbrace{\text{smallness}}_{\uparrow}$

"factor"

type of compactness.

compactness
= partition (———)
was finite on
closed manifolds.

section of a bundle.

this bundle is f.d. $\Leftarrow$

1D. $\Leftarrow$

In Pure alg:

Small = f.d. sep.

Type I = central simple.
= Azumaya.