Operators and (higher) categories in QFT II

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Today: 2 textbook examples of operators in QFT.

Both examples will be lagrangian — presented by path integral. Not every QFT can be presented by a path $\int$.

The same QFT can have multiple inequiv. path $\int$ descs.

Input for "lag QFT" is: sheaf of "fields" and "action".

Compact boson: in nD

fields on M^n = functions $X^n \xrightarrow{f} S^1 = \mathbb{R}/\mathbb{Z}$.

action $$S(f) = \frac{1}{2}\int_{X^n} df \star df$$

$$= \frac{1}{2}\int_{X^n} g^{\mu\nu}\,\partial_\mu f\,\partial_\nu f\,\sqrt{g}$$

Classical EOM $\equiv$ critical points $S(-)$ = harmonic functions

Contemplate

$$Z(X^n) = \int_{f: X \to S^1} \mathcal{D}f \quad \exp(-S(f))$$

in true QFT, want i in place of -1, should still suppress high energy fields, but more slowly.

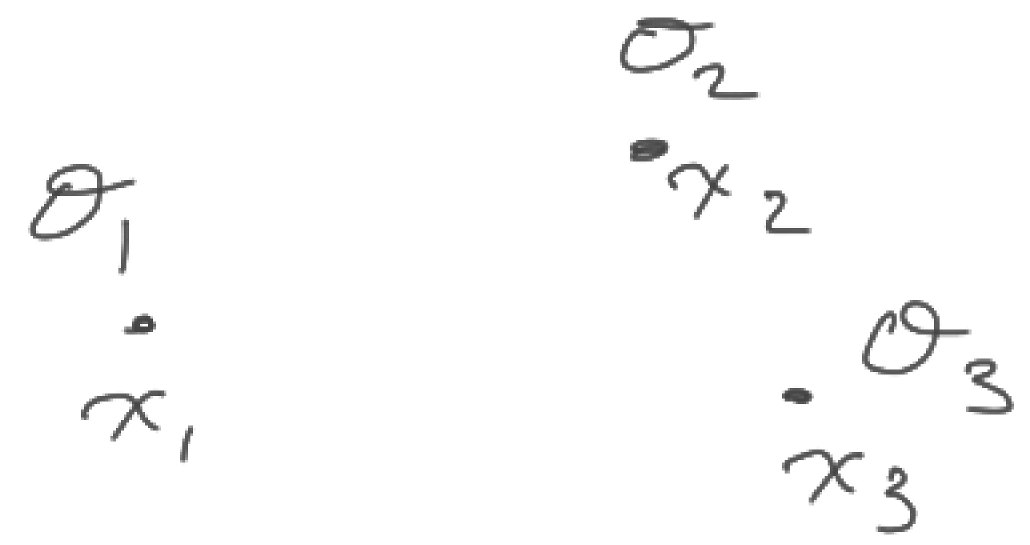
Integrals like this should converge because:

- {low energy fields} is f.d. compact manifold. (X is compact)
- {high energy fields} are suppressed in the $\int$.

We get a collection of local operators, namely polys in jets of a field.

e.g. $\sin(2\pi f^{(0)})(x)$, $\underset{f^{(1)}}{df}(x)$

Σ not all the ops, and there are redundancies



$$\mathcal{O}_i = \mathcal{O}_i(f^{(0)}, f^{(1)}, \dots)$$

$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \dots \rangle :=$ expectation value for the product of these numbers

$$= \frac{1}{Z(X)} \int_{\text{fields}} \mathcal{D}f \;\; \mathcal{O}_1(f)(x_1) \dots e^{-S(f)}.$$

$S(f) := \frac{1}{2} \int_X df \star df$. This action is invariant under $f \mapsto f + \varepsilon$ $\quad \varepsilon \in \mathbb{R}$ $\quad \varepsilon \ll 1$.

$\mathcal{D}f$ invariant under $f \mapsto f + \varepsilon g$ $\quad \forall \varepsilon \in \mathbb{R}$ $\quad g : X \to \mathbb{R}$.

is supposed to be trans. inv on $[X, S^1]$

pick $x_0 \in X^n$, pick $g(x)$ supported near x_0.

$O_1(x_1) \dots O_k(x_k)$ say have some other operators, inserted away from $\mathrm{supp}(g)$.

$$\int \boxed{\mathcal{D}f \; O_1(f)(x_1)\, O_2(f)(x_2) \cdots \exp\left(-\frac{1}{2}\int_X df \star df\right)}$$

$$= \int \underbrace{\mathcal{D}(f+\varepsilon g)}_{=\;\mathcal{D}f} \; O_1\underbrace{(f+\varepsilon g)}_{=\;O_1(f)(x_1)}(x_1) \cdots \exp\Big(-\underbrace{\frac{1}{2}\int_X d(f+\varepsilon g)^{\star 2}}_{=\;-S(f) - \varepsilon\int_X df \star dg + \dots}\Big)$$

$= S(f+\varepsilon g)$ only has dg – no $g^{(0)}$

$$= \int \boxed{} \cdot \left(1 - \varepsilon \int_X df \star dg + \dots\right)$$

i.e.

$$\int_{\text{fields}} \mathcal{O}_1 \cdots \mathcal{O}_k \cdot \Big(\int_X \overbrace{df \star dg}^{\pm (\star df)\, dg} \Big) \cdot \exp(-S(f)) = 0$$

$$\underbrace{\int_X df \star dg}_{= \pm \int g(x) (d \star d f)}$$

$$= \int_{x \in X \setminus \{x_1, \dots x_k\}} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) [\star df](x) \rangle \underbrace{dg(x)}_{\text{arbitrary exact 1-form away from other insertion pts.}}$$

<u>Conclusion</u>:

$\star df$ is a $(n-1)$-form valued operator in fact valued in closed $(n-1)$-forms.

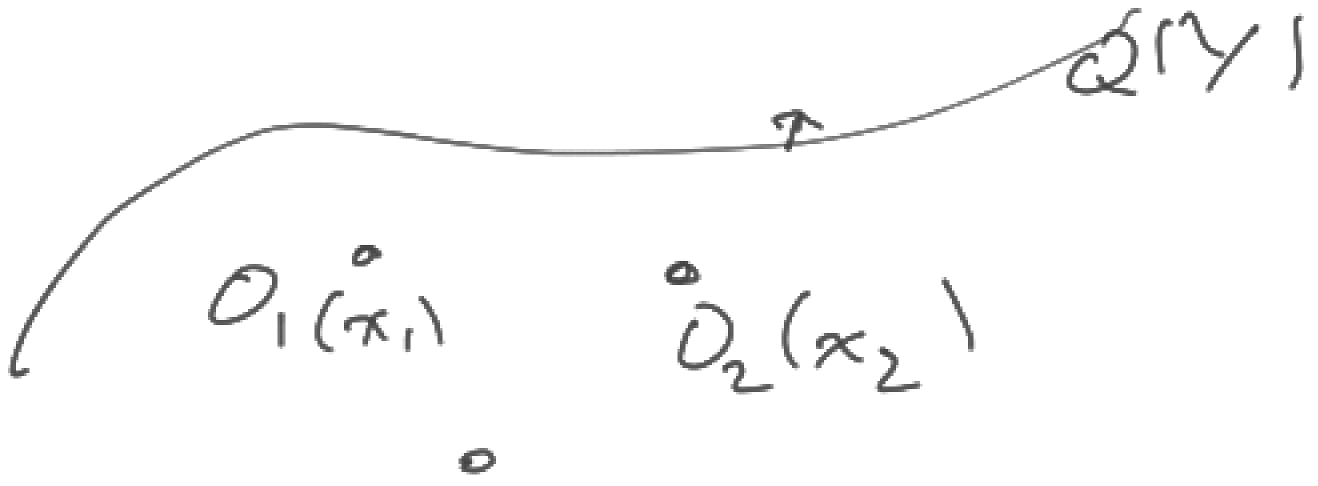
i.e. $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \, (d \star d f)(x_0) \rangle \equiv 0$.

$\star\, df$ is a (closed) $(n-1)$-form valued operator.

$\rightsquigarrow$ extended operator

$$\begin{array}{ccc} Y^{n-1} & \longmapsto & \displaystyle\int_Y \star\, df =: Q(Y) \\ \downarrow & & \\ X^n & & \end{array}$$

closure of $\star\, df$ means $Q(Y)$ is topological



$\langle Q(Y) \cdot O_1(x_1) \dots O_k(x_k) \rangle$

depends only on

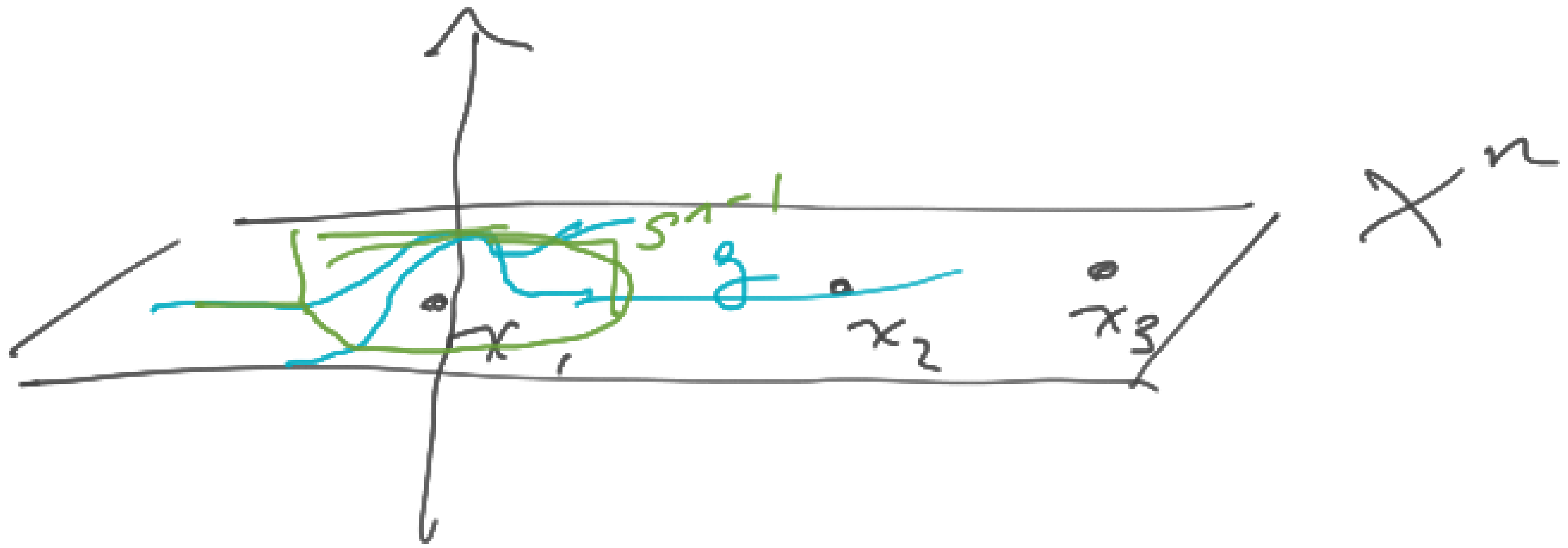
$[Y] \in H_{n-1}(X^n \setminus x_1, \dots x_k)$

I want to convince you that $Q(-)$ is not trivial.

Repeat the path $\int$ analysis where

dg is supported away from $x_1 \ldots x_4$

but $g(x_1) = 1$ $\quad g(x_2) = \ldots = g(x_4) = 0.$



$\mathcal{O}_1(f)(x)$

$= \mathcal{O}_1(f^{(0)}_{(x)}, f^{(1)}_{(x)}, \ldots)$

$\mathcal{O}_1(f)(x_1) + \varepsilon \frac{\partial \mathcal{O}}{\partial f^{(0)}}(x_1) \cdot g(x_1)$

$\quad \overset{\shortparallel}{} \qquad\qquad\qquad\qquad \overset{\shortparallel}{1}$

$\langle \mathcal{O}_1(f)(x_1)\, \mathcal{O}_2(x_2) \ldots \rangle = \langle \mathcal{O}_1(f + \varepsilon g)(x_1) \ldots$

$\qquad (1 - \varepsilon \int_X dg * df) \rangle$

$\langle \frac{\partial \mathcal{O}_1}{\partial f^{(0)}}(x_1) \cdot \ldots \rangle = \langle \mathcal{O}_1(x_1) \cdot \int_X dg * df \rangle$

Punchline:

$Q(S^{n-1})$

$$= \frac{\partial \mathcal{O}_1}{\partial f^{(0)}} \bullet x_1$$

i.e. this topological operator Q is the infinitesimal generator of the symmetry $f \mapsto f + \varepsilon$.

$\star df$ is called the "Noether current".

Q is the corresponding conserved quantity
$\shortparallel$
topological

Another example: Maxwell theory.

$\shortparallel\shortmid$

$\mathbb{R}/\mathbb{Z}$ gauge theory.

Fields = $U(1)$-bundle w/ connection

locally = 1-form A_μ

modulo $A_\mu \mapsto A_\mu + \partial_\mu \lambda$

$\lambda : X^n \to \mathbb{R}/\mathbb{Z}$.

i.e. can take prestack $\dfrac{\Omega^1_{\mathbb{R}}(-)}{d(\Omega^0_{\mathbb{R}/\mathbb{Z}}(-))}$

stackify. = $\{U(1)$ bundles w/ $\nabla\}$.

$$S(A) = \frac{1}{2} \int_{X^n} dA \star dA$$ $F_{\mu\nu} = \partial_{\{\mu} A_{\nu\}}$

exactly same analysis:

$\star dA$ is a closed op-valued $(n-2)$-form.

$\rightsquigarrow$ top. operator $Q(Y^{n-2}) = \int_Y \star dA$

$U(1)$ is abelian, so stack $\mathbb{B}U(1)$ is a gp.

acts on $\{U(1)\text{-bundles w/ } \nabla \text{ on } X\} = C^\infty$ maps $X \to \mathbb{B}U(1)$

this Q is the inf. generator.

$e = \exp(2\pi i\, -)$

compact boson		Maxwell	
$\exp(2\pi i\, f)$	0D	$\exp(2\pi i \int_{L'} A)$	1D
$\int_{L'} df$ top!	1D	$\int dA$ top!	2D
$Q = \int_{Y^{n-1}} \star df$ top!	$(n-1)$D	$\int \star dA$ top!	$(n-2)$D
flux / 't Hooft operator	$(n-2)$D	't Hooft operator	$(n-3)$D

don't commute

sym. generators

"wilson operator"
charge = +1

Exercise:

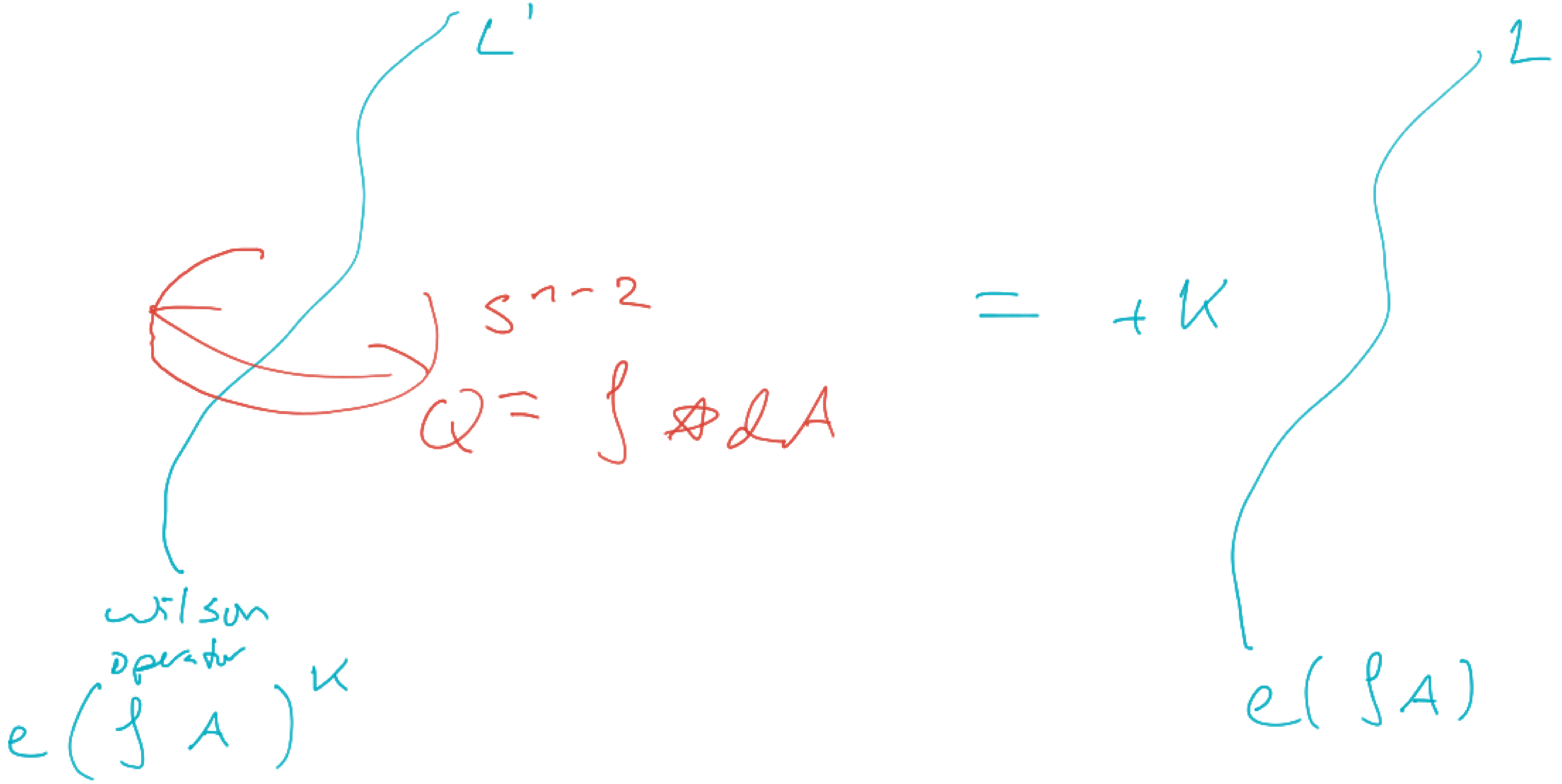
2D compact boson conformal field thy.

4D Maxwell is CFT and has this EM duality auto.

$$[Q, e(f)] = +1 \cdot e(f)$$

i.e. "$e(f)$ has charge 1 under Q"

i.e. $\subset X^n$



I told you that a way to construct operators in a Lag QFT is evaluating fct of field.

if $\mathcal{O}_1 \ldots \mathcal{O}_k$ are of this type, then

$$\langle \mathcal{O}_1 \ \ldots \ \mathcal{O}_k \int_{L'} d f \rangle$$

$$= 0 \quad \text{if } \partial L' = \emptyset .$$

Another type of operators in a Lag QFT are ones whose insertion changes the domain of path integration.

In compact boson case, Maxwell thy

$\exists$ $(n-\cancel{2}\,3)$ D operator R (not top.)

effect of inserting $R(Y^{n-2})$

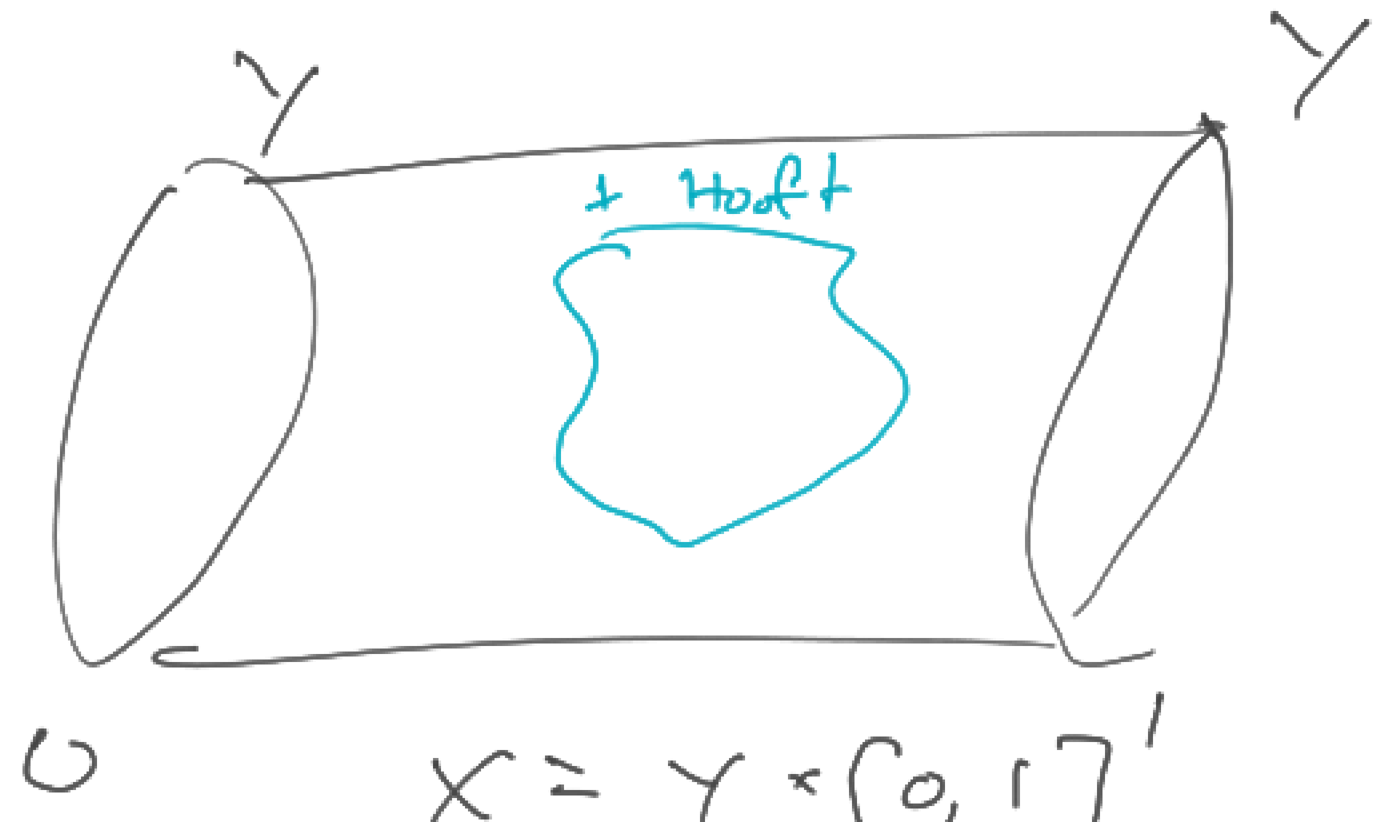
changes $\displaystyle\int_{X \to \mathbb{R}/\mathbb{Z}} \rightsquigarrow \int_{(X-Y) \to \mathbb{R}/\mathbb{Z}}$

and around Y have winding $+1$ and prescribed asymptote as you approach Y.

In a 2^{nd} order Lagrangian thy e.g. compact boson,

$$\mathcal{H}(Y^{n-1}) = L^2(\text{fields}(Y^{n-1}))$$

$$= L^2(C^\infty(Y, \mathbb{R}/\mathbb{Z}))$$



$X = Y \times [0,1]$

In 3D, compact boson $\underset{\text{as QFT}}{\cong}$ Maxwell.

Moral: "being a gauge thy" is not (always) a meaningful property of a QFT.

It is a property of a Lag presentation of a QFT.

State-operator correspondence:

$$\mathcal{O} \longmapsto \text{functions} = L^2(\text{Fields}(S^{n-1}))$$

in the TFT case

Local Op $\longrightarrow \mathcal{H}(S^{n-1}_R)$

$$\uparrow s$$

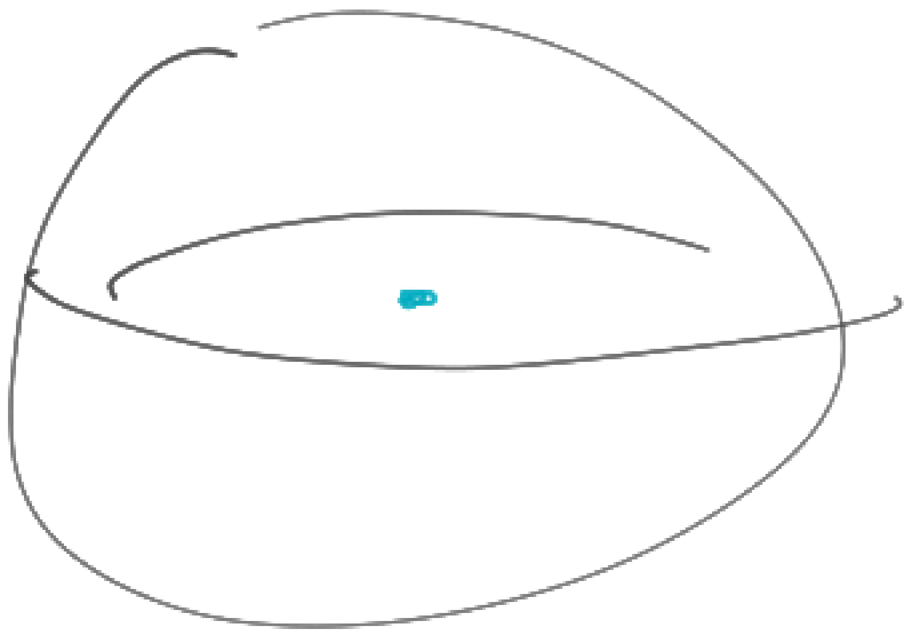
$$\mathcal{H}(S^{n-1}_{R'}) \quad R' < R$$

$$\uparrow s$$

$$\lim_{R \to 0} \mathcal{H}(S^{n-1}_R)$$

S^{n-1}_R

S^{n-1}



$$\mathcal{O} \longmapsto \left(\partial f \longmapsto \int \mathcal{O} \cdot e^{-S(f)} \right)$$

fields on S^{n-1}

fields f w/ $\partial f = \partial f$

L^2(boundary fields)

End(Q) is monoidal $(n-1)$-cat.

nD QFT

in TFT case where I know a defn

$\cup|$

{local operators} $= H(S^{n-1})$.