Operators and (higher) categories in QFT III

Goal: Categorical condensation.

We have some nD QFT

$\rightsquigarrow$ { extended operators } (dimensions $0, \dots, n$)

$\cup$ |

{ topological operators }

These form an n-cat. $\mathcal{C}$

"pointed" $1 \in \mathcal{C}$

k-morphisms = codim-k operators

objects = "spacetime-filling operators"

$\hookrightarrow$ distinguished one "vacuum",

"invisible"

I'm not aware of any proposed mathematics describing "alg structure" on these.

Defn: An n-category is an $(\infty,1)$-cat enriched in $(n-1)$-categories. [c.f. Schommer-Pries's Notre Dame lectures]

In particular, Given objects X, Y, $\hom(X,Y)$ is an $(n-1)$-cat.

In particular, $\mathbb{1} \in \mathcal{C} \rightsquigarrow \hom(\mathbb{1},\mathbb{1}) =: \Omega\mathcal{C} \ni \mathrm{id}_{\mathbb{1}}$

get to multiply depending on order on a line

$\Omega\mathcal{C}$ is a monoidal $(n-1)$-cat.

$\Omega^k\mathcal{C}$ is a k-monoidal $(n-k)$-cat

$\mathrm{ob}(\Omega^k\mathcal{C}) =$ k-endos of $\mathbb{1}$.

$= (n-k)$-dim operators.

" E_k " ← multiplication depends on order on a k-dim space.

local operators $\sim \mathcal{H}(S^{n-1})$

$\cup\!\!\mid$

top local operators — is a vector space

i.e. a "linear 0-cat"

[Line ops

$\cup\!\!\mid$

top line operators] — Linear 1-categories.

E.g.: Look at G-gauge thy e.g. YM thy

G is a compact gp.

Fields: G-bundles w/ ∇. (Locally $\partial_\mu + A_\mu$)

YM action $= \int \langle F \otimes F \rangle$

Given a f.d. rep'n $V : G \to GL(n)$, there is

a Wilson operator

line operator $L \mapsto \boxed{\text{Pexp} \int_L} V(A_\mu)$

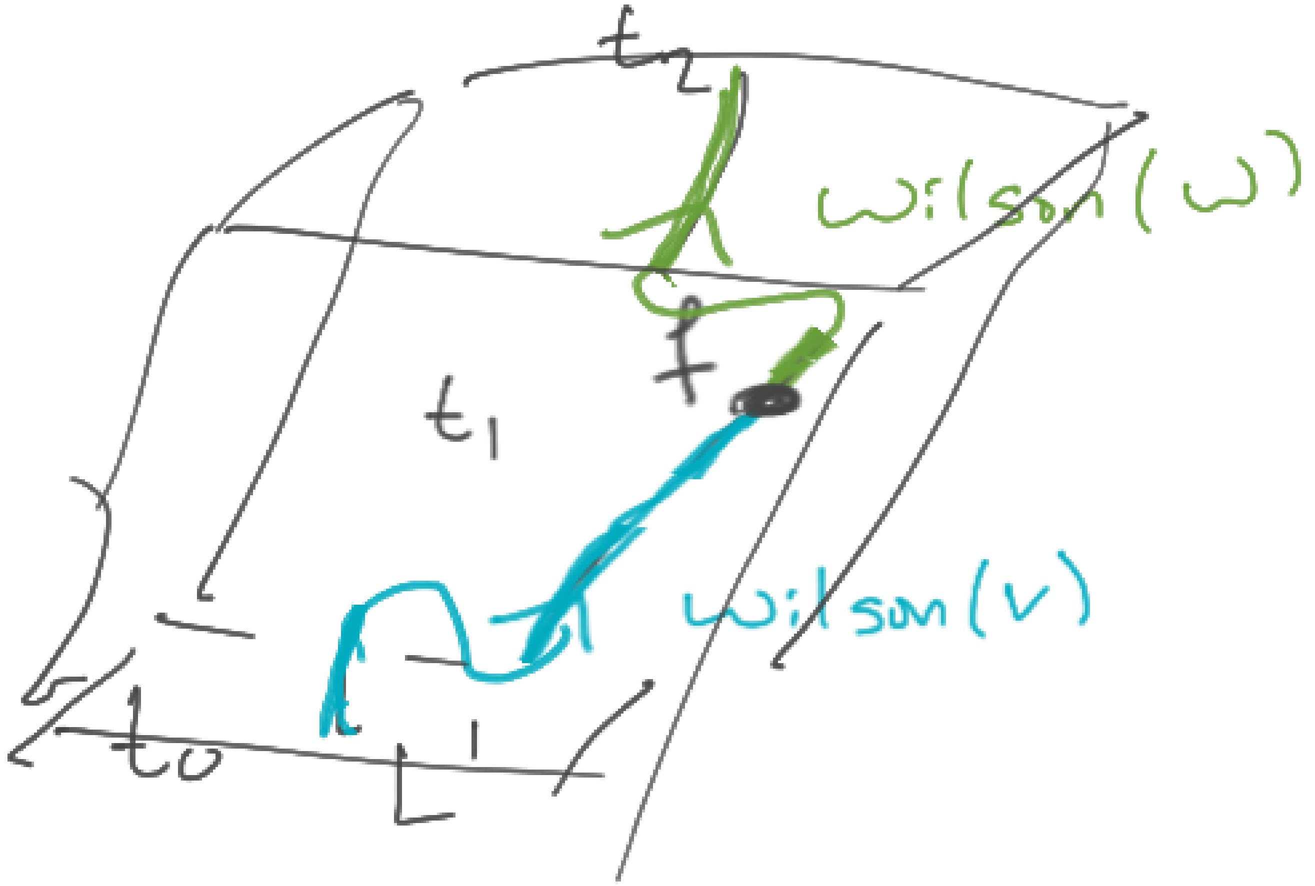
$\Omega^1 \otimes \mathfrak{gl}(n)$

Pexp$\int$ means:

- cut up L into little pieces.
- Π in order.

Rep(G) $\rightsquigarrow$ line operators in G-gauge thy.
i.e. things you can insert along curves.

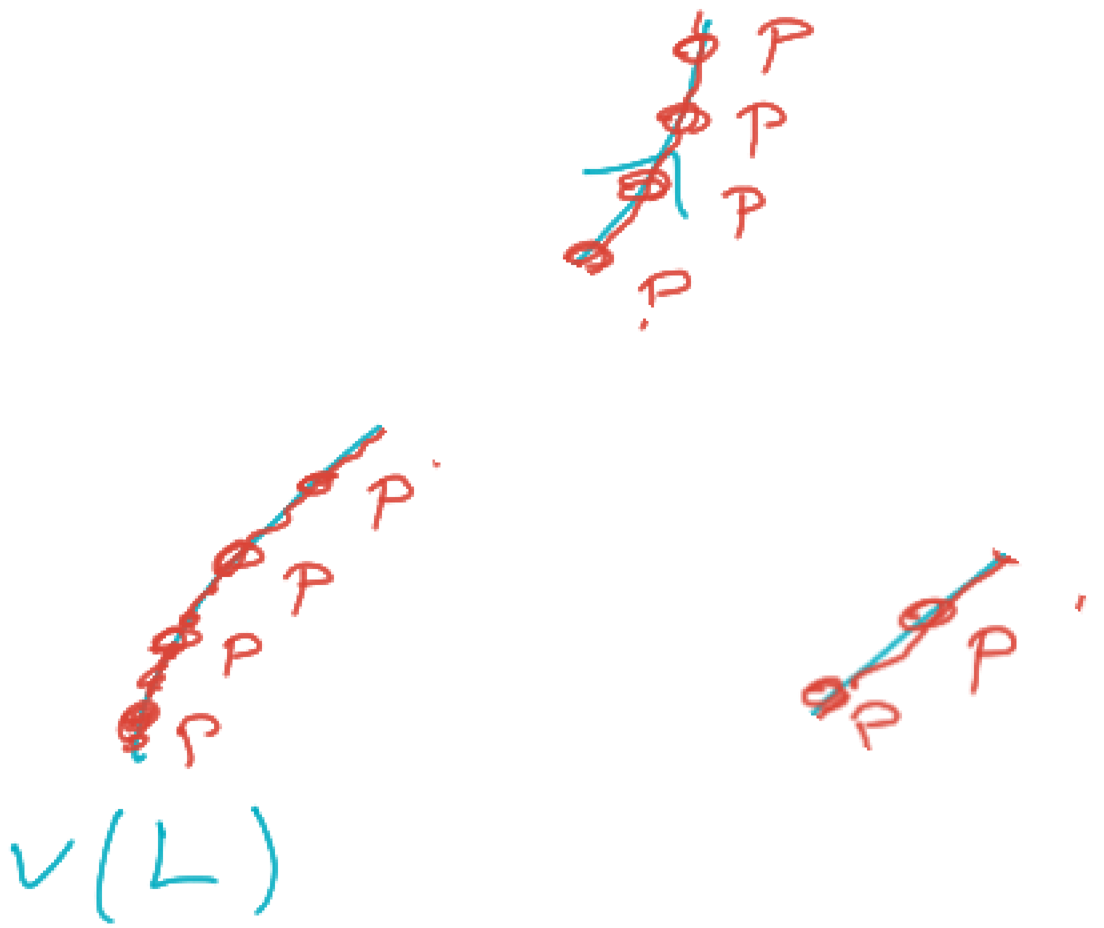
Suppose $f: V \to W$ is a homomorphism in Rep(G).



$$\underbrace{\overbrace{\mathrm{pexp}\left(\textstyle\int W A_\mu\right) \circ f \circ \underbrace{\mathrm{pexp}\textstyle\int V(A_\mu)}_{\in GL(V)}}^{\circledast}}_{\in \mathrm{Mat}(V,W)}$$

t is a parameterization of L'

This composition $\circledast$ does not depend on value of t_1.
i.e.  is a topological interface between these line ops.



Pick some line operator,

e.g. $\mathrm{Wilson}(V) \equiv V$

pick topological interface $p : V \to V$.

ask it to be idempotent $p^2 = p$.

i.e. choice of $\mathrm{im}(p) \oplus V$

In case Wilson op, p came from G-submodule, we just produced Wilson op for that submodule.

- locations of where I insert p don't matter.
- $\lim\limits_{\text{density of insertions} \to \infty} \left(\quad \right)$ exists. and is a line op.

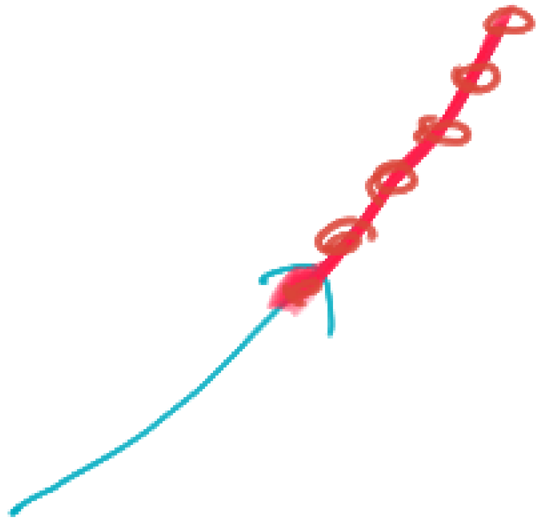
reason for thinking of dense insertions is so that I can compute locally.

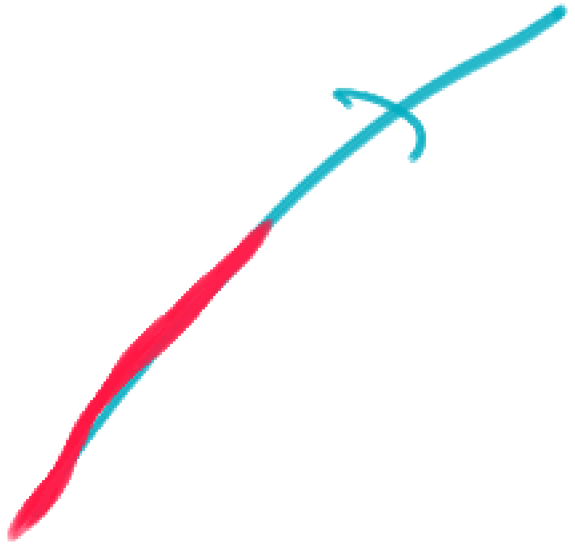
$$V = \overset{1}{\mathrm{im}(p)} \oplus \overset{0}{\cdots} \quad \text{evalues of } p.$$

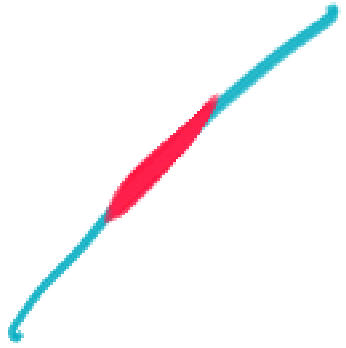
This procedure of populating a V-line w/ dense config of p $\underbrace{\text{defects}}_{\substack{\parallel \\ \text{interface}}}$ condenses V onto $\mathrm{im}(p)$.

$$V \supset \mathrm{im}(p)$$

$$\underbrace{\mathrm{im}(p) \hookrightarrow V \twoheadrightarrow \mathrm{im}(p)}_{\substack{\parallel \\ \mathrm{id}_{\mathrm{im}(p)}}}$$



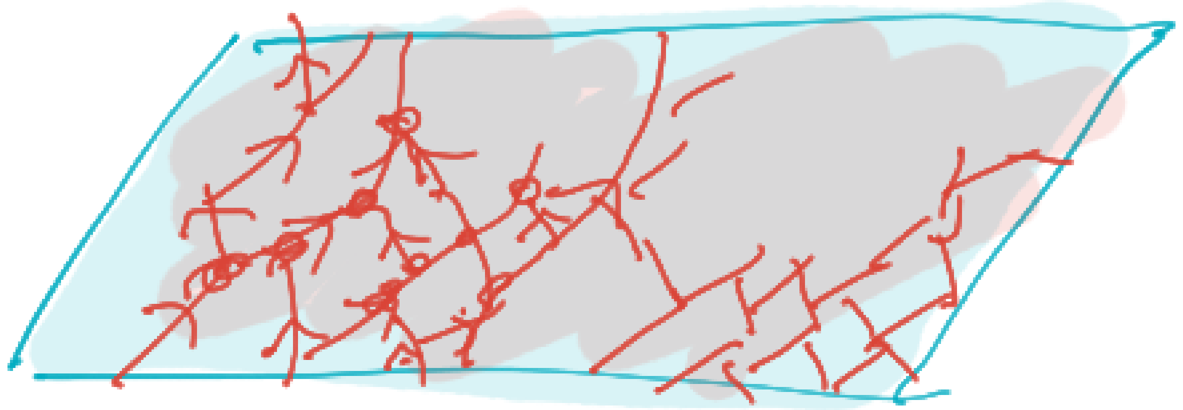




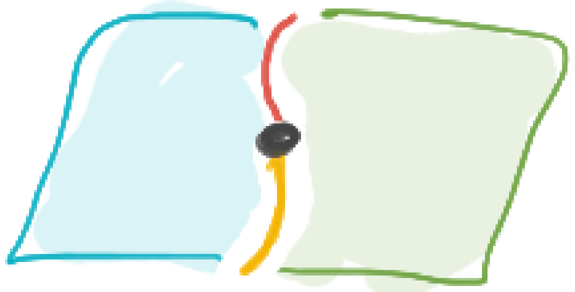
$V \twoheadrightarrow \mathrm{im}(p)$ $\qquad$ $\mathrm{im}(p) \hookrightarrow V$ $\qquad$ $= p.$

Let's try to do the same thing for 2D operators

$V \leftarrow$ same 2D op.



e.g. $V = \int F$



in gauge thy.

in 4D, $V = \int \star F$

$\exists$ 2-cat:

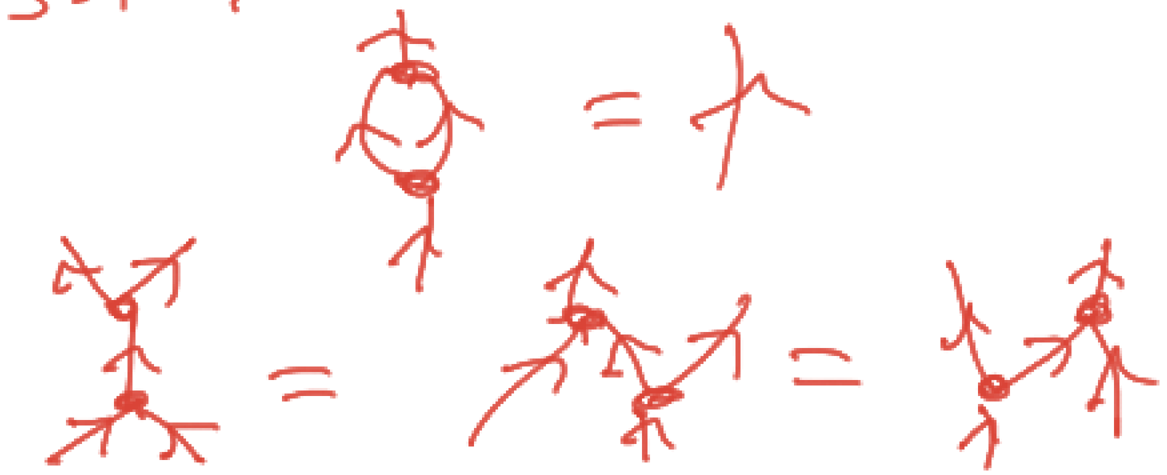
- 2D operators
- top. interfaces (1D)
- top. junctions of interfaces. (0D)

Q: what can we populate it with to get another surface operator?

A: It suffices to choose: top. interface $p : V \to V$

and

s.t.



It suffices to choose: top. interface

$p : V \to V$

and

hom(p^2, p)

hom(p, p^2)

s.t.

special / separated

AKA: non unital Frob. algs

AKA: 2-idempotent

i.e. $p^2 \cong p$.

i.e. p is idemp.

N.B: If  are an ISO.

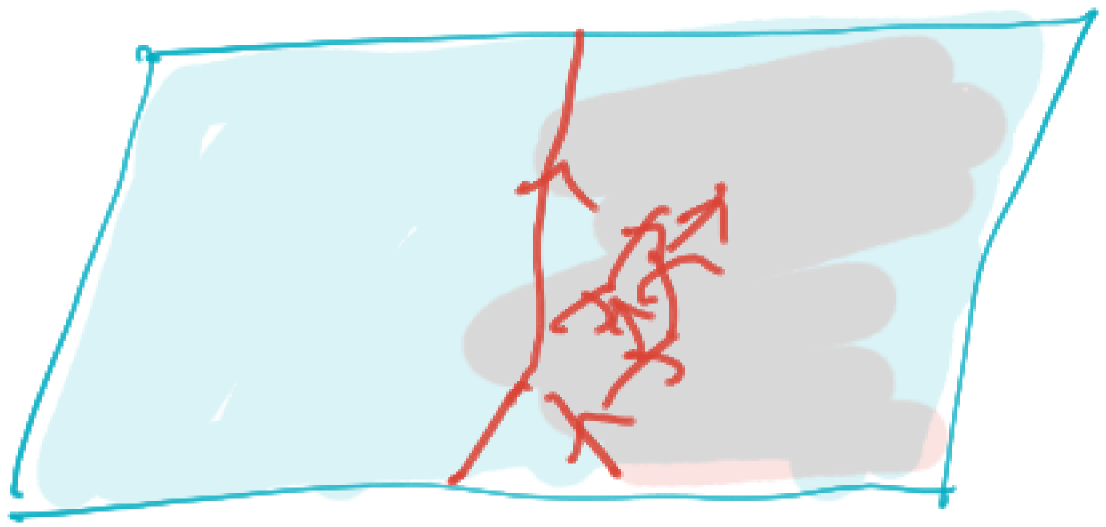
Defn: In a 2-cat, a 1-morphism $p: V \to V$ with [ ] is a condensation monad.

monad $\equiv$ alg obj in an endo cat (in a 2-cat)

Ex: ✪ $\Rightarrow$  is assoc.

i.e. 

and Y is coassoc.

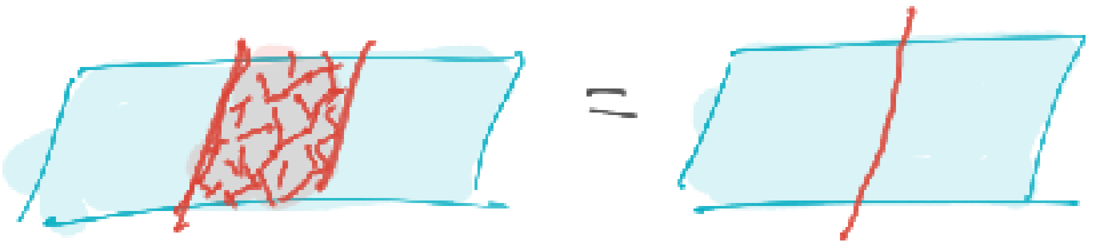


get a top. interface

$V \longrightarrow \mathrm{im}(p)$

$\quad\quad\quad\quad\;\; \|$

result of fusing V w/ dense config of these 1- and 2-mors.

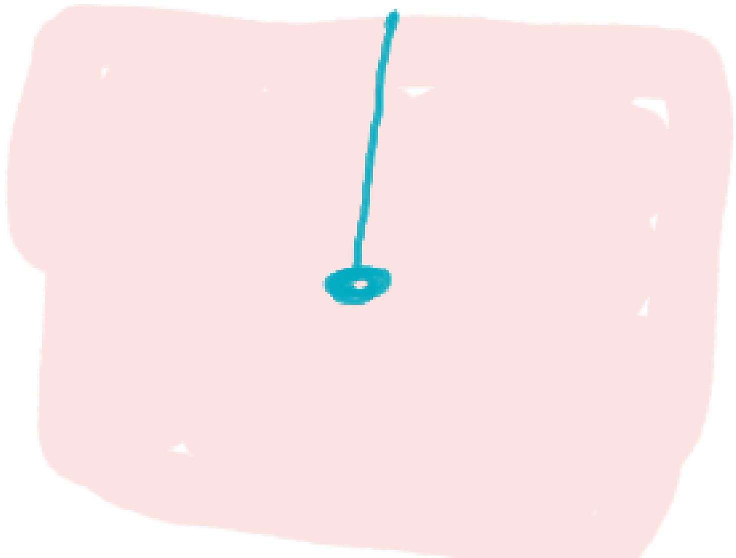


p

$\not\cong \mathrm{id}_{\mathrm{im}\,p}$

$\mathrm{im}(p)$ "cs" V "$\twoheadrightarrow$" $\mathrm{im}(p)$

$\mathrm{hom}(\mathrm{id}_{\mathrm{im}(p)}, \mathrm{comp}) \ni$



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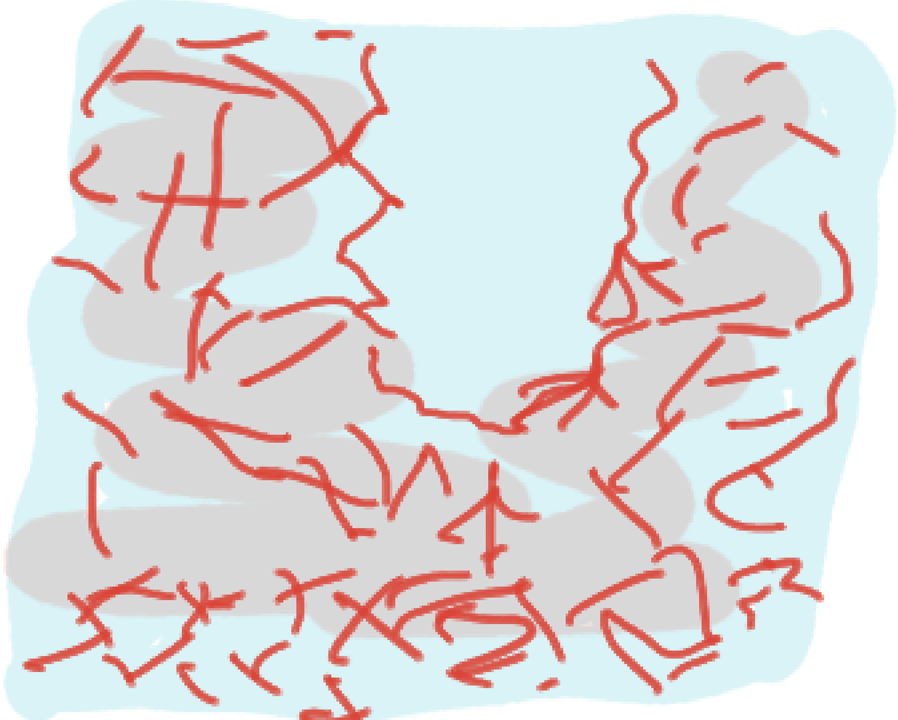


In a 1-cat

$p : V \circlearrowleft$ s.t. $p^2 = p$

$\Updownarrow$

$$\begin{tikzcd} im(p) = W \arrow[r, shift right] & V \arrow[l, shift right] \end{tikzcd}$$

s.t. $W \circlearrowleft = id_W$.

In a 2-cat

2-idempotent on V

$\Updownarrow$

$$\begin{tikzcd} & V \arrow[dl, bend right, "f"'] \\ im(p) = W \arrow[ur, bend right, "f^*"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} W \arrow[loop, "f^* f"] \end{tikzcd} \quad \underset{\Leftarrow}{\overset{ev_f}{\Rightarrow}} \quad id_W$$

↑ choose splitting of ev_f.

s.t. $= id_{id_W}$

Language: Any time you can build higher-dim op. from lower-dim ops in some systematic way, the higher dim op is a "descendant".

- $\mathrm{im}(p)$ is a <u>condensation descendant</u> of $p \in \mathrm{End}(\omega)$
- in compact boson, $\underbrace{(\exp) \int df}_{\text{descended from } f.}$ $\quad f : X^n \to \mathbb{R}/\mathbb{Z}.$

Prop: Suppose V, ω are 2D operators w/ a top'l interface $f : V \to W$

s.t. (0) $\{\text{top. } \underline{\text{local}} \text{ ops on } \omega\} = \Omega \mathrm{End}(\omega) = \mathbb{C}$ "ω is simple"

(1) $\{\text{top. lines on } \omega\} = \mathrm{End}(\omega)$ is <u>s.s.</u>

enough for $\mathcal{C}$ to be projective.

"Schur's Lemma"

Prop: Suppose V, W are 2D operators w/ a top'l interface $f: V \to W$, $f \neq 0$.

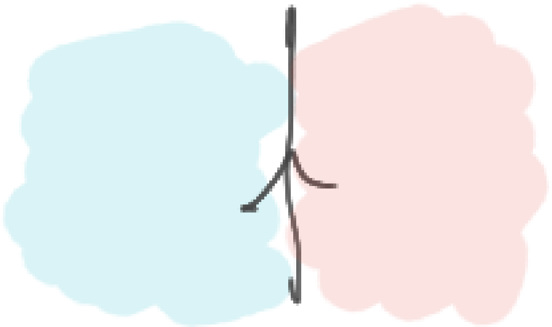
s.t. (0) $\{\text{top. local ops on } W\} = \Omega\,\mathrm{End}(W) = \mathbb{C}$ "W is simple"

(1) $\{\text{top. lines on } W\} = \mathrm{End}(W)$ is $\underline{\underline{\text{s.s.}}}$ (enough for $\mathcal{C}$ to projective.)

Then W is a cond. descendant of V.

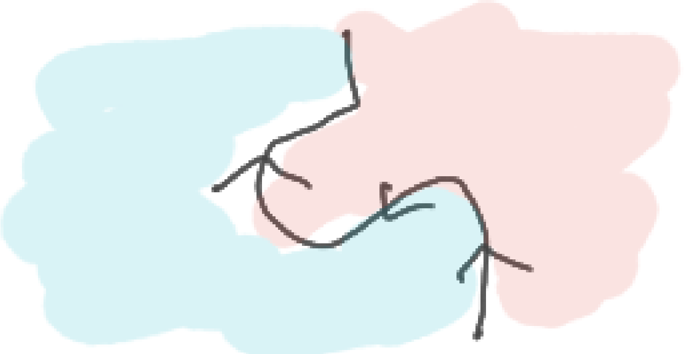
i.e. $\exists$ 2-idempotent $p \in \mathrm{End}(V)$ s.t. $W = \mathrm{im}(p)$.

Pf:



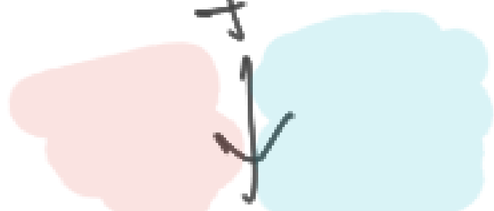
V f W

$\rightsquigarrow$



produce

f^*



Look at $f \circ f^*: W \to W$. it has nonzero 2-morphism

$f \circ f^* \Rightarrow \mathrm{id}_W$ ← simple object in s.s. cat.

<u>Defn:</u> An n-cat is <u>semisimple</u> if

- additive ($+$ of n-mors, $\oplus$ of $(<n)$-mors)
- all k-idempotents should have images (at all levels)
- all $(<n)$-mors should have adjoints
- all 1-cats of $(n-1)$-morphs should be s.s.

---

in any RT thy e.g. CS thy

- all surfaces are descendants $\Leftarrow$ selected by cond. algs in the MTC of lines.
- two surfaces are iso $\equiv$ condensation algs are Morita equiv. $\equiv$ mod cats for the MTC

A good defn of holomorphic CFT

(2D QFT all ops are holo in $z = x+iy$)

is VOA V s.t. $\mathrm{Rep}(V) = \mathrm{Vec}$.

E_8 Lie gp $\qquad \mathbb{M}$

Famous examples: $E_{8,1}$, $V^\natural$.

$\{\text{line ops}\} = \{\text{top lines}\}$ $\quad \infty$-dim

$\cup|$

ob $\{\text{inv. top lines}\} = \mathrm{Aut}(\mathrm{CFT})$

For any $\underline{\underline{\text{compact}}}$ connected gp,

$\exp : \overset{\text{Noether currents}}{\mathfrak{g}} \to \overset{\text{autos.}}{G}$ is a surjection.

In E_8 case, inv. lines are all Noether descendants.

in monster case ???

$$W \xhookrightarrow{\text{conf. embedding}} V$$

should be able

~~rational~~ VOA $\qquad\qquad$ holomorphic VOA

$\wr$

~~MTC~~ $\qquad$ $\mathrm{Rep}(W) \cong Z(\mathcal{F})$ $\qquad$ ∞-dim $\otimes$

∞-dim $\beta\otimes$ cat $\qquad\qquad$ some ~~fusion~~ cat.

Given $W \hookrightarrow V$, $\quad$ V is a W-module

in fact a E_2-alg obj. in $\mathrm{Rep}(W)$.

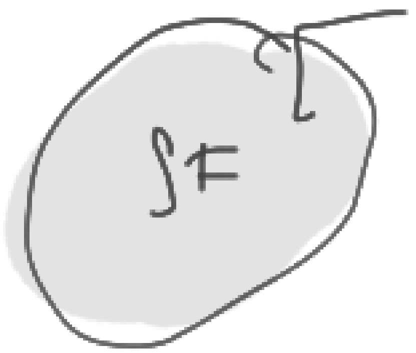
$\mathcal{F}$ = its cat of mod. objects.

Take W = subalg of V generated by T.

Maxwell $\qquad F = dA$

$$\exp\left(\int F\right) = \text{triv.}$$

$\|$



$=$

$= \exp \int A$

"$\int F$ is $\mathbb{Z}$-valued"

*** **Exercise:** Interpolate between exp(Stokes' Thm) and cond. descent.
Along the way, invent non-topol, non-inv descent.