Operators and (higher) categories in QFT IV

Last time: some intuition: whatever an nD QFT is, from its data you can extract an n-category $\mathcal{C}$ of "topological operators". E.g. $ob(\mathcal{C})$ = spacetime-filling top'l ops.

This $\mathcal{C}$ is:

- $\mathbb{C}$-linear and has $\oplus$s.
- rigid i.e. all k-morphisms $0<k<n$ have both adjoints.
- Karoubi complete i.e. all higher idempotents have images
  "condensation complete"   condensation monad " higher idempotents
- pointed i.e. distinguished object $1 \in \mathcal{C}$ = invisible operator.

☆ $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(1) \ni id_1$

- $X \twoheadrightarrow Y$ if X carries a higher idemp. w/ image Y.

"X condenses onto Y"

If $X \twoheadrightarrow Y \twoheadrightarrow Z$ then $X \twoheadrightarrow Z$.

$\Omega^{n-k-1}\mathrm{End}(X)$

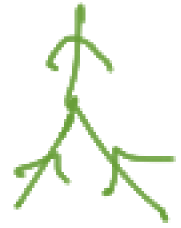
- X is simple if the "endo"-top-morphisms of X $= \mathbb{C}$

↑ some k-morphism

- $\mathcal{C}$ is pointed connected if $\forall X \in ob(\mathcal{C})$, $1 \twoheadrightarrow X$.

and 1 is simple.

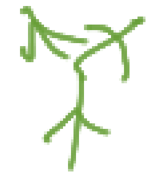
Ex: Assoc algs in $Bim(A)$ $\longleftrightarrow$ alg homs $A \to B$.



Algs and bimodules = ALG

$A \in ALG$ $\quad$ $End(A) \cong Bim(A) \ni P$

(If unital alg) $\quad$ P is an sep'l alg ~~ext.~~ of A.



If $1 \twoheadrightarrow X$ $\forall X$, then

$$\mathcal{C} \equiv \mathrm{Kar}(\underbrace{\text{full subcat on } 1}_{\substack{\shortparallel \\ \text{1 pt delooping of } \otimes\ (n-1)\text{-cat} \\ \Omega\mathcal{C}}})$$

So $(\Omega\mathcal{C}, \otimes)$ knows all the data of $\mathcal{C}$.

Defn: $\mathcal{C}$ is semisimple if ($\circledast$ from slide 1) and all 1-cats of $(n-1)$-mors are s.s. (i.e. every obj is a finite $\oplus$ of simples)

* Exercise: If $\mathcal{C}$ is pointed connected and invisible $(n-1)$-morphism is projective in $\Omega^{n-1}\mathcal{C}$ and all 1-cats of $(n-1)$-mors are l.f.a. then $\mathcal{C}$ is s.s.

Higher Schur's lemma (Douglas – Reutter)

if $f : X \longrightarrow Y$, X, Y simple
$\not\cong 0$

and if $\mathcal{C}$ is s.s. then $X \twoheadrightarrow Y$
and $X \twoheadleftarrow Y$

and so "$\hom(X, Y) \neq 0$" is an equiv relation on simples.

In 1-cat, this is the reln $\cong$, but not in n-cat.

X and Y are "Schur-connected".

** Exercise: Study higher Schur's lemma w/ s.s.

$\pi_0 \mathcal{C}$ = Schur components of $\mathcal{C}$

= equiv classes of $\twoheadrightarrow$ for simples.

$\rightsquigarrow \quad \pi_k \mathcal{C} := \pi_0 \Omega^k \mathcal{C}, \quad k \leq n-1$

the homotopy sets of $\mathcal{C}$.

[I'm always imagining $\mathcal{C}$ is s.s]

[I'm mostly imagining $1 \in \mathcal{C}$ is simple]

If $\mathcal{C}$ is pointed connected

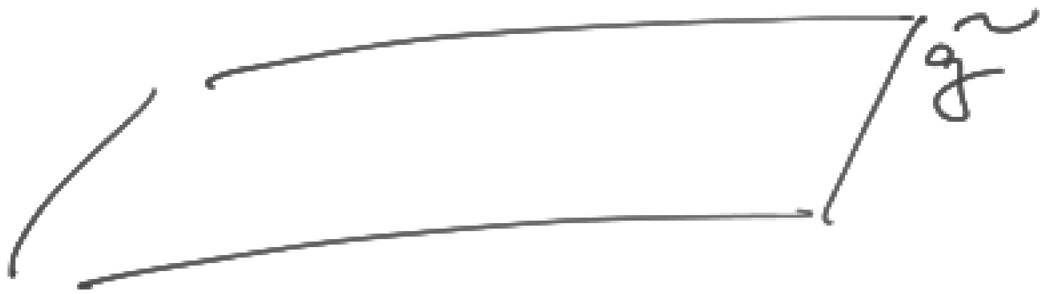
i.e. $\pi_0 \mathcal{C} = *$, 1 is simple.

then $\mathcal{C}$ has the same data as $\Omega \mathcal{C}$

I'll call it <u>fusion</u> if $|\pi_k \mathcal{C}| < \infty \ \forall k$.

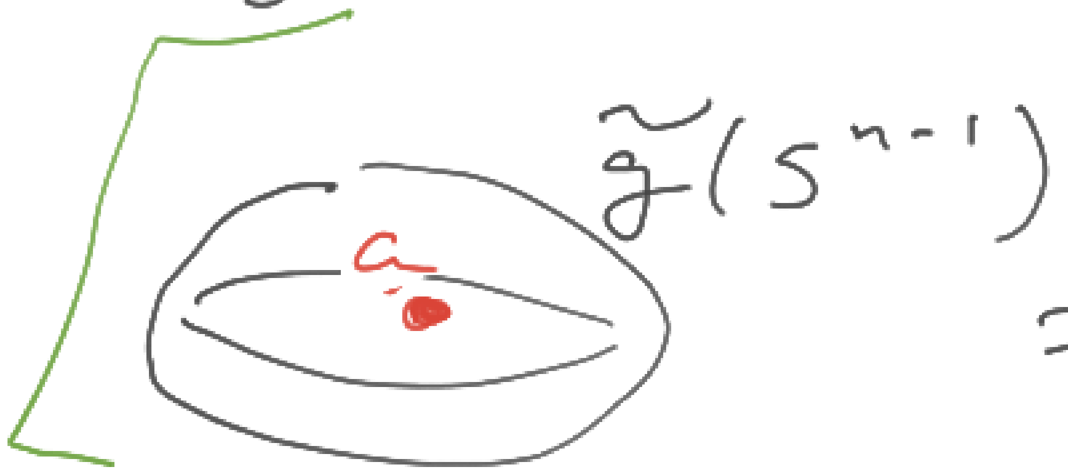
Top operators $\longleftrightarrow$ "non-invertible symmetries"

If $g \in \mathrm{Aut}(Q)$ Noether promises you a top'l invertible $(n-1)$D operator.



If you have a local operator a

conj a by $\tilde{g}$.

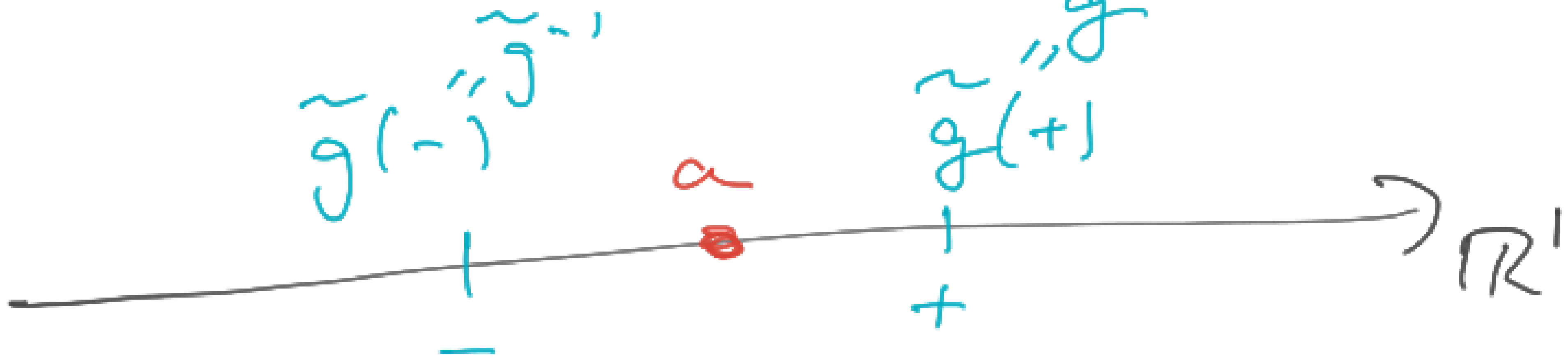


$= \; g(a)$

Compare: $n = 1$.

$$A^\times \twoheadrightarrow \mathrm{Aut}(\text{matrix alg } A)$$

$$\tilde{g} \longmapsfrom g$$



$$g[a] = g^{-1} a g$$

To understand this picture, I'd like to remind you about the embedded cob. hyp.

$$\underset{n\text{-cat}}{\mathcal{C}} \rightsquigarrow \Omega^k \mathcal{C} \ni X.$$

rigid k-monoidal $(n-k)$-cat.

I'd like to insert X on an $(n-k)$-manifold

$$\Sigma^{n-k} \hookrightarrow \mathbb{R}^n$$ (or any other framed n-manifold)

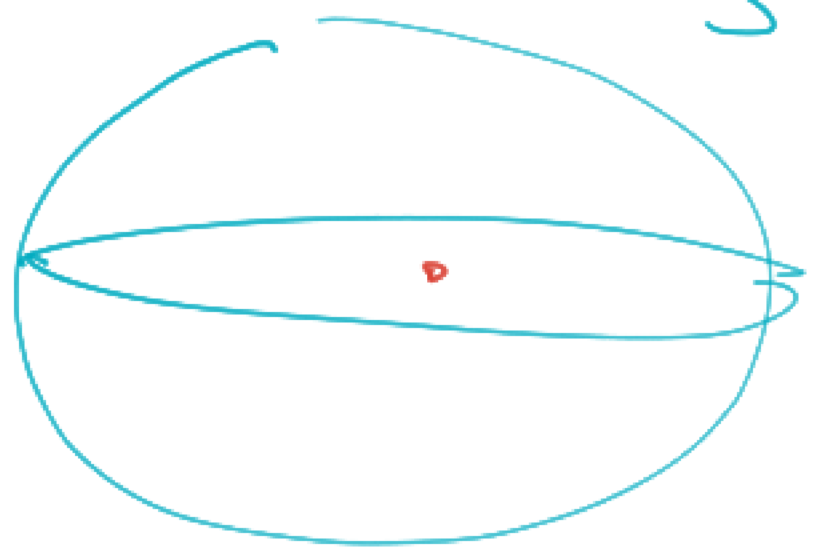
You need Σ to be

- tangentially framed $\quad T_\Sigma \xrightarrow{\tau, \sim} \mathbb{R}^{n-k}$
- normally framed $\quad N_\Sigma \xrightarrow{\nu, \sim} \mathbb{R}^k$
- $\nu \oplus \tau \cong$ trivial

If X is invertible, then you only need τ stably.
$\tau := \text{ambient} - \nu$.

I want to insert X along

$S^{n-k} = \partial D^{n-k+1}$



This sphere has a stable framing so enough to insert invertibles.

Not enough framing to insert dualizables that aren't invertible.

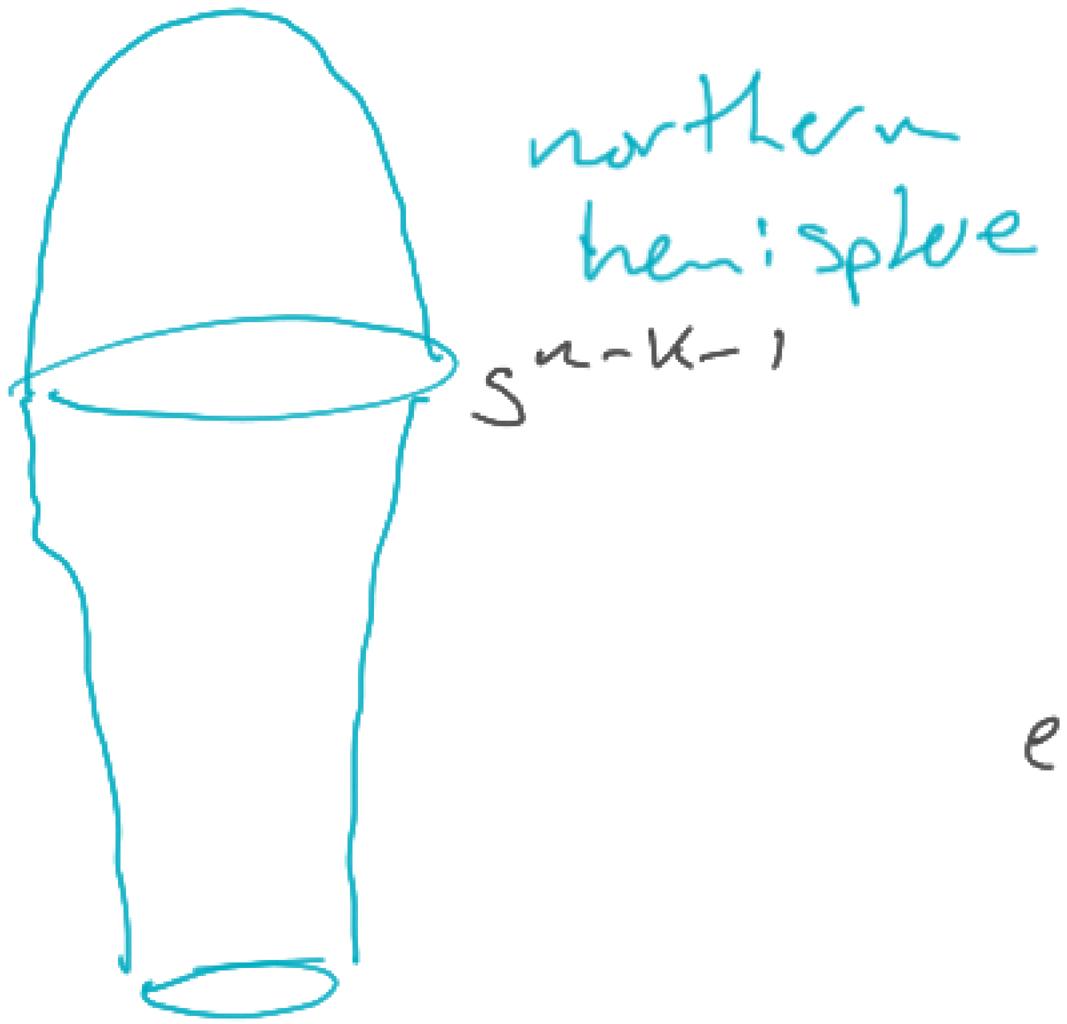
 $\longmapsto$ evaluation morphism for X.

if X is inv, then ev_X is inv,

so  $:= ev_x^{-1}$

$X \in \Omega^k \mathcal{C}$ — Assume $\mathcal{C}$ s.s. and pointed connected

and X is simple.



$\neq 0$

$\int_{\text{hemisphere}} X = \uparrow$ n-morphism from inv. line] — simple object in $\Omega^{n-1}\mathcal{C}$.

$\int_{\text{equator}} X =$ line operator you get by wrapping X around a cylinder

$\uparrow$

S^{n-k-1} framed in the way that extends over northern hemisphere

X being simple $\Longleftrightarrow$ $\int_{\text{equator}} X = \underbrace{\text{inv.}}_{\text{appears w/ mult } 1.} \oplus \ldots$

Conclusion: If $\mathcal{C}$ s.s., simple unit, and x simple

$\exists$ unique splitting of ev_x

i.e. a unique "southern hemisphere"

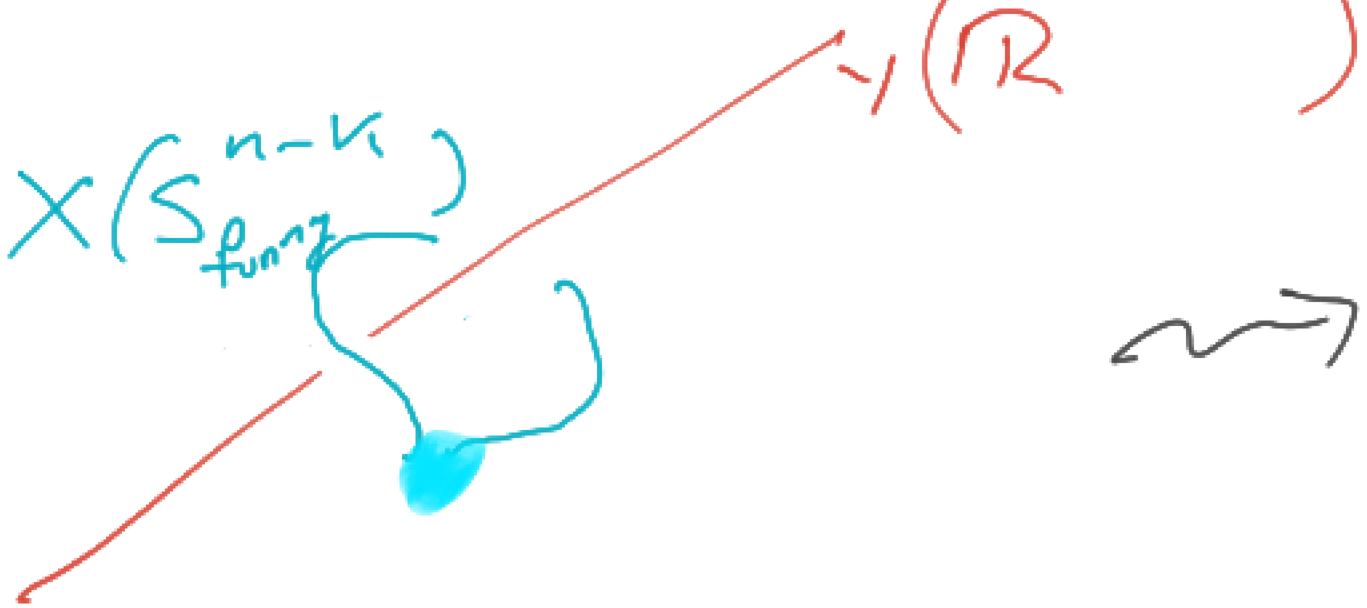
s.t.

$= 1.$

Pick $X \in \Omega^k \mathcal{C}$, $Y \in \Omega^m \mathcal{C}$

simple

this is a picture of conjugating Y by X. $\rightarrow$



$\rightsquigarrow$ $m+k+\{0,1,-1\}$-morphism on Y

i.e. element of $\Omega^{\cdot\cdot}\,\mathrm{End}(Y)$

Linguistics: A k-form sym is an element of $\pi_k \mathrm{Aut}(QFT)$

$\xleftrightarrow{\text{Noether}}$ $(n-k-1)$-dim inv. ops i.e. inv elts of $\Omega^{k+1}\mathcal{C}$.

n-cat $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{A} = \Omega\mathcal{C}$ $\otimes$ (n-1)-cat.

$\mathcal{A}^\times$ grouplike $\otimes$ (n-1)-gpoid

$\circlearrowright$ conj.

$\mathcal{A}$

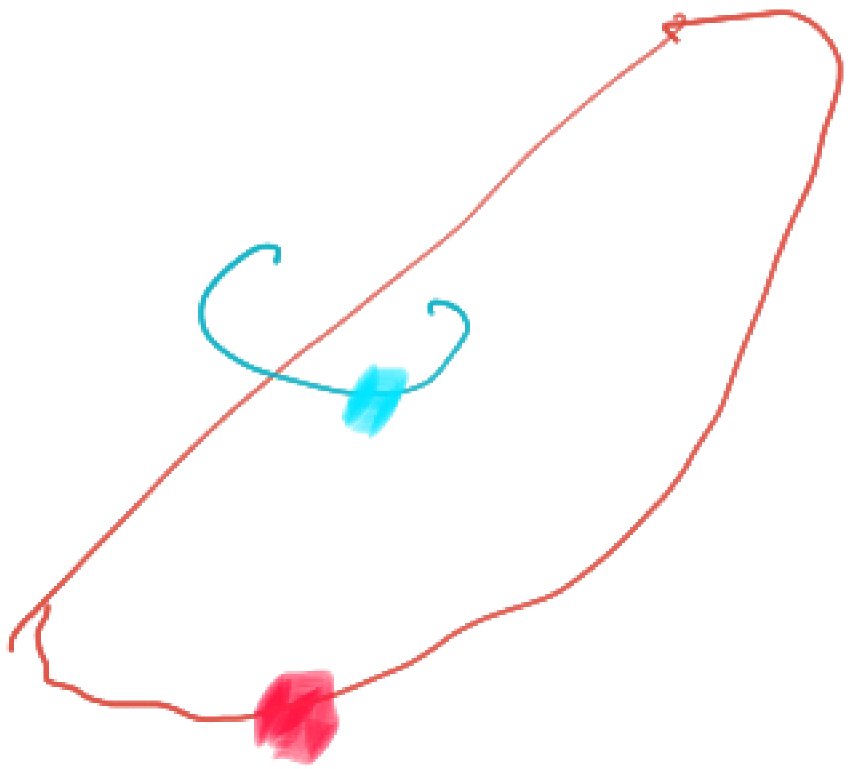
k-morphisms in $\mathcal{A}$ / iso.

$$\begin{array}{ccc} \overbrace{\pi_k \mathcal{A}^\times}^{\cong} & \times & m\text{-morphisms in } \mathcal{A} / \text{iso} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & & Y \end{array}$$

$\longrightarrow$ m+k-morphism in $\mathcal{A}$ /iso.

X $\qquad$ Y

Smth simple, of comp. dim.



$\rightsquigarrow$ top-endo of Y $\cong \mathbb{C}$

Lemma: This number only depends on Y and X up to Schur component.

In other words:

$$\left\{\begin{array}{c}\text{simples} \\ \text{in} \\ \Omega^k \mathcal{C}\end{array}\right\} \times \left\{\begin{array}{c}\text{simples} \\ \text{in} \\ \Omega^{n-k-1} \mathcal{C}\end{array}\right\} \longrightarrow \mathbb{C}$$

factors thru

$$\underset{\substack{\cup \\ X}}{\pi_k \mathcal{C}} \times \underset{\substack{\cup \\ Y}}{\pi_{n-k-1} \mathcal{C}} \longrightarrow \mathbb{C}$$

"e-value of action of X on Y"!

"Framed S-matrix"! "Whitehead bracket"!

$\mathcal{C}$ n-cat $\rightsquigarrow$ $\Omega\mathcal{C}$ $\otimes$ (n-1)-cat.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathcal{A}$

<u>Noether-type</u> <u>axiom</u>: $\mathcal{A}$ should be morita-inv.
for TQFTs.

$\Leftrightarrow$ $\mathcal{A}$ should be f.d.
w/ $Z(\mathcal{A}) = (n-1)\mathrm{Vec}$.

unwritten
<u>Thm (JF-Reutter)</u> $\mathcal{A}$ is morita inv
iff Fusion $(\pi_k \mathcal{A} < \infty)$
and
$$S : \underbrace{\pi_k \mathcal{A} \times \pi_{n-k} \mathcal{A}}_{\neq 1} \to \underbrace{\mathbb{C}}_{\sim 1}$$
all nv. $\forall k$.

Noether axiom

$\Rightarrow$ $\exists$ exact triangle

$$\begin{tikzcd} \mathcal{A}^\times \arrow[rr] & & \mathrm{Aut}(\mathcal{A}) \arrow[dl] \\ & (n\mathrm{Vec})^\times \arrow[ul] & \end{tikzcd}$$

$\leftarrow$ a spectrum has lots of triv. homotopy gps.

$$\begin{tikzcd} \text{simples in } \mathcal{A} \arrow[rr] & & \text{"non-inv autos"} \arrow[dl] \\ & \cdots \arrow[ul] & \end{tikzcd}$$