Operators and higher categories in QFT V

Goal

Preamble:

An operationally defined QFT in $(n+1)D$ is something like $\otimes$ n-cat

$\mathcal{A} = \{$ operators of dim $0, \dots n$
i.e. codim ≥ 1 $\}$

$\cup$

and chosen $\frac{\partial S}{\partial g^{\mu\nu}} \sim T_{\mu\nu}$ some local op w/ ...

axiom 1: $\mathcal{A}$ is "Type I" i.e. "$\mathrm{Bim}(\mathcal{A}) = (n+1)\mathrm{Hilb}$"

axm 2: "compactness"

$\mathcal{A}$

subalg gen by $T_{\mu\nu}$

F.A.

commutator
$\equiv$ top'l ops

Expect: • bicommutant thm.
• Peter-Schur-Weyl th

Data of bicommutant pair inside Type I
$\equiv$ two pieces & $\otimes$ equiv
$\mathrm{Bim}(\{T_{\mu\nu}\}) \simeq \mathrm{Bim}(\text{top'l ops})$

Last time: In an (n+1)D QFT there is a rigid $\otimes$ n-cat of top'l operators. Under some semisimplicity assumptions, you should think of them as "noninvertible" or "categorical" symmetries.

Goal for today: What does it mean to "gauge" some categorical symmetry?

Warm up: Complicated description of gauging action of a finite gp.

Input: homomorphism

$$\begin{tikzcd} & & \text{inv. top ops} \arrow[d] \\ & G \arrow[ur, "\mu"] \arrow[r] & \mathrm{Aut}(\text{ops}) \\ \text{finite gp} \arrow[ur] & & \end{tikzcd}$$

If you just had $G \to \mathrm{Aut}(\text{ops})$ the obstruction to μ is some "anomaly"!

Look at "norm element"

$$N = \bigoplus_{g \in G} \mu(g)$$

In $(0+1)D$
$N = \sum \mu(g)$.
$N^2 = (\# G) \cdot N$
so $\frac{N}{\# G}$ is idep.

Claim: This N carries naturally the structure of an n-cat'l idempotent.
In char 0. aka condensation alg.

Pf: The gp hom $G \xrightarrow{m} \mathcal{A}^\times \equiv \begin{pmatrix} \text{top'l} \\ \text{ops} \end{pmatrix}^\times$

is the same data as a $\otimes$ functor

$$\underset{g \in G}{\bigoplus} g \in n\mathrm{Vec}[G] \longrightarrow \mathcal{A}$$

$= \mathrm{fibre}^*(1)$ it is an alg.

$=$ formal sums $\bigoplus V_g \cdot g$ with each $V_g \in n\mathrm{Vec}$.

$n\mathrm{Vec} := \Sigma\, (n-1)\mathrm{Vec}$

$\uparrow$ Karoubi complet'n of 1-object delooping.

$(\Sigma \dashv \Omega.)$

$$n\mathrm{Vec}[G] \xrightarrow{\text{fibre}} \underset{1}{\overset{\cup}{n\mathrm{Vec}}}$$

$\mathrm{fibre}^*: n\mathrm{Vec} \to n\mathrm{Vec}[G]$ is lax $\otimes$.

$$\begin{array}{c} n\mathrm{Vec}[G] \\ \big\downarrow \text{fibre} \\ n\mathrm{Vec} \end{array}$$

extends to a condensation $\begin{array}{c} n\mathrm{Vec}[G] \\ \big\downarrow \\ n\mathrm{Vec} \end{array}$

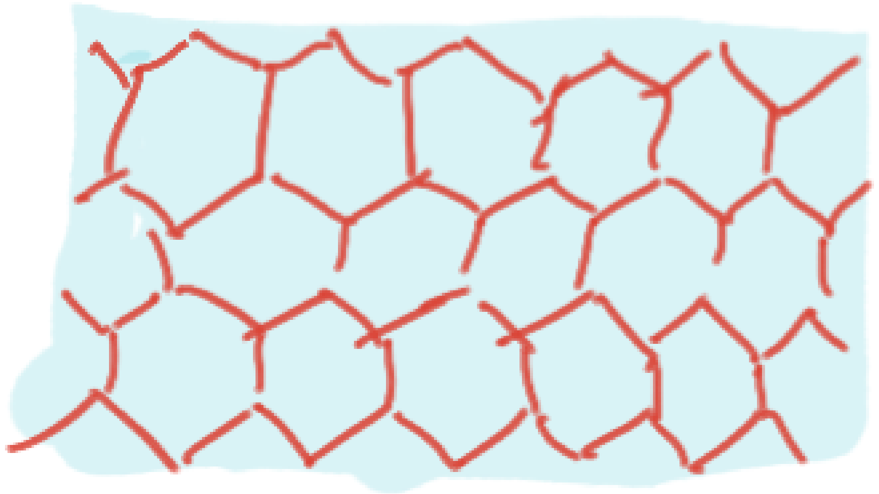
in cat of $n\mathrm{Vec}[G]$-modules.

("Triv rep of G is projective")

i.e. $\exists$ "idempotent" in $n\mathrm{Vec}[G]$ whose image is $n\mathrm{Vec}$. This idemp is $\bigoplus_{g \in G} g$.

So $M(\bigoplus_{g \in G} g)$ is also idemp. $\square$.

Take our $(n+1)$D QFT and "condense" this norm elt N, i.e. flood spacetime with a network of N interfaces.

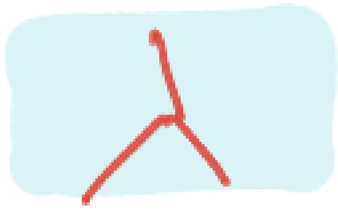


original QFT Q

$N = \bigoplus_{g \in G} \mu(g)$

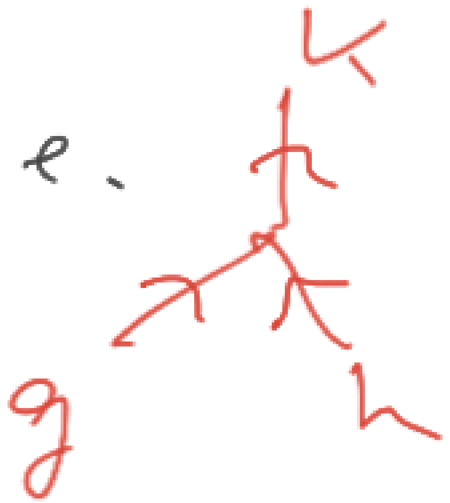
(Recall: "higher idempt" "condensation alg" is what you need so that this flood doesn't depend on which fine network you use. It works for any cellulation of spacetime.)

So result of condensation is a sum over all ways to decorate each wall of the cellulation by an elt of G.



where walls meet, you put either mult or comult in Frob alg $\bigoplus \mu(g)$

These enforce the labels to be compat

i.e. 

only contributes to sum $k = gh$.

In other words this flood is a Σ over (trivialized G-bundles on spacetime

in the open cells, but this is harmless because of the normalization factor on some top morphism defs

Punchline: Result of condensing this condensible alg $N = \bigoplus_{g \in G} \mu(g)$

is the gauged thy $Q /\!/ G^{\mu}$.

Q = orig. QFT. μ = anomaly cancellation data.

Take any $(n+1)$D QFT.

spacetime filling id op. $\mathbb{1}$

A codim $= 1$ op is an interface from $\mathbb{1} \to \mathbb{1}$.

$\{\text{operators of codim} \geq 1\} = \mathrm{End}(\mathbb{1})$

$\sim \otimes$ n-cat $\qquad \sim (n+1)$-cat

What are the operators of $Q/\!/G$?

In a 1-cat, $X \circlearrowleft_p$ $p^2 = p$,

and $\mathrm{End}(\mathrm{im}(p)) = \{ f \in X \text{ s.t. } pf = f = fp. \}$

$$\begin{array}{ccc} & & fp \\ \text{not hom} \quad \text{functor} & \uparrow & \uparrow \mathbb{1} \\ & \{\mathrm{End}(X)\} & f \end{array}$$

In the gauge thy case, some formulas are:

$$\underset{\substack{\parallel \\ A/\!/G}}{\mathrm{ops}(Q/\!/G)} = \mathrm{End}_A\Big(\underset{\substack{\parallel \\ \mathrm{ops}(Q)}}{A} \underset{n\mathrm{Vec}[G]}{\otimes} \mathbb{1} \Big)$$

"If" $A = \mathrm{End}(\mathcal{H})$

$G \circlearrowright \mathcal{H}$

$$A /\!/ G = \left(A \underset{n\mathrm{Vec}[G]}{\otimes} \mathbb{1} \right)^G = \mathrm{End}(\mathcal{H}/G)$$

<u>C.f.:</u> <u>Ham reduction:</u>

moment map

$$T^*X = \underset{\text{symp}}{M, \omega} \xrightarrow{\mu} \mathfrak{g}^*$$

Ham action by Lie alg $\mathfrak{g}$.

$X \circlearrowleft \mathfrak{g}$

$\updownarrow$

Poisson map $\mathrm{Sym}(\mathfrak{g}) \to C^\infty(M)$

$\subset$ Hopf

$$\left(C^\infty(M) \underset{\mathrm{Sym}(\mathfrak{g})}{\otimes} \mathbb{1} \right)^{\mathfrak{g}} = C^\infty\left(\overbrace{\mu^{-1}(0)/\mathfrak{g}}^{M/\!/\mathfrak{g} \,=\, T^*(X/\mathfrak{g})} \right)$$

QFTs are something like sheaves

$G \circlearrowright$ some sheaf $\mathcal{E}$ over M^{n+1},

$\Updownarrow$

a way of extending $\mathcal{E}$ to a sheaf on $\mathrm{Bun}_G(M)$

$\Updownarrow$

" coupling $\mathcal{E}$ to maps $M \xrightarrow{P} \mathbb{B}G_\delta$ ".

<u>Gauging</u>: $\int$ over choice of P.

$Z(M, P)$ might not be a number. It might change by $\times$ scalar under gauge trans.

i.e. It might be a section of $\begin{matrix} L \\ \downarrow \\ \mathrm{Bun}_G(M) \end{matrix}$.

Say $\mathcal{Z}$ is a fusion n-cat. E.g. $\mathcal{Z} = n\mathrm{Vec}[G]$.

<u>Input</u>: • "Inner action" of $\mathcal{Z}$ on our QFT,

i.e. $\otimes$ map $\mathcal{Z} \longrightarrow$ top'l ops.

• fibre functor, i.e. $\otimes$ map $\mathcal{Z} \to n\mathrm{Vec}$.

<u>E.g</u>: If $\alpha \in Z^{n+2}(G; \mathbb{C}^\times)$, I can write a twisted gp al'g $n\mathrm{Vec}^\alpha[G]$.

Inner action of $n\mathrm{Vec}^\alpha[G]$ is an action of G w/ anomaly α.

If $\alpha = \partial\beta$, then you get a fibre functor.

when $n \gg 0$, fibre functors "=" triv of α in an appropriate extraordinary coh.

*** <u>Exercise:</u> For any fusion n-cat $\mathcal{G}$, any fibre: $\mathcal{G} \longrightarrow n\mathrm{Vec}$, selects $n\mathrm{Vec}$ as the image of a cond. alg $\in \mathcal{G}$.
$$\overset{\shortparallel}{\mathrm{fibre}^*(1)}$$

$\approx$ all fusion n-cats are separable in char 0.

<u>Warn:</u>
- not true $n=1$ and char $\neq 0$. (even over a perfect field)
- $n=1$, char $=0$, pf uses statements about positivity of e-values of integer matrices.

D-S-SP, using ÉGNO

$\mathcal{G} \xrightarrow{\mu} \mathcal{A}$

which idemps in $\mathcal{A}$ arise in this way?

Conj: Any fusion $\mathcal{G}$, fibre works.

$\Downarrow$

idempotent who knows the fibre functor $\mathcal{G} \to n\mathrm{Vec}$ $\longmapsto$ μ(this idemp)

so condense it.

Otherwise: just use those ($\mathcal{G}$, fibre) with separability property.

$\mathcal{A} /\!\!/ \mathcal{G}$ = result.

at the level of operators, result = $(\mathcal{A} \otimes_{\mathcal{G}} n\mathrm{Vec})^{\mathcal{G}}$.

This is what condensed matter physists call anyon condensation:

Take MTC $\mathcal{M}$. braid $\otimes$ 1-cat.

$A \in$

sep com etale alg object.

(generate a braided fusion cat $\mathcal{B}$ = gen by A)

s.t. $A \leftrightarrow$ fibre

$$\begin{array}{c} A \in \mathcal{B} \\ \downarrow \\ \text{fibre} \end{array}$$

$\mathcal{M}$

$\downarrow$

$\Sigma\mathcal{M}$ $\otimes$ 2-cat

$=$

Kar(one obj. delooping)

$=$

{$\mathcal{M}$-module cats}.

$\mathcal{B} \longrightarrow \mathcal{M}$ braided $\otimes$.

$$\begin{array}{ccc} \Sigma\mathcal{B} & \longrightarrow & \Sigma\mathcal{M} \\ \downarrow & & \\ 2\text{Vec} = \Sigma\text{Vec} & & \end{array}$$

...

1-form sym.

$\downarrow$

A finite ab gp. $\longrightarrow$ ab. bosons in $\mathcal{M}$.

$\mathcal{M}$ MTC

i.e.

$\Sigma \mathrm{Vec}[A]$ ~~braided~~ 2-fusion $\longrightarrow \Sigma\mathcal{M}$.

$\downarrow$ braided $\otimes$

$\Sigma \mathrm{Vec}$

$$\Omega\left(\Sigma\mathcal{M} /\!/ \Sigma\mathrm{Vec}[A]\right)$$

$\parallel$

gauging the 1-form sym. $=$ result of usual anyon cond.

fusion n-cat

$$\begin{tikzcd} \mathcal{G} \arrow[rr] \arrow[d, "\text{fibre}"] & & \mathcal{A} \\ n\mathrm{Vec} & & \\ \mathcal{G}' := \mathrm{End}_{\mathcal{G}}(n\mathrm{Vec}) \arrow[u] \arrow[rr] & & \mathcal{A}/\!/\mathcal{G} \end{tikzcd}$$

$$\mathcal{A}/\!/\mathcal{G} = \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{G}} n\mathrm{Vec}) \curvearrowleft \mathcal{G}'$$

$\pi_0 \; n\mathrm{Vec}[G] = G$

$\pi_{>0} = * \quad \ldots$

$\pi_{n-1} \; n\mathrm{Rep}(G) = \{\text{irreps of } G\}$

$\pi_{<n-1} = *.$

$\mathcal{G}'$ is another fusion n-cat Morita equiv to $\mathcal{G}$

$$(\mathcal{A}/\!/\mathcal{G})/\!/\mathcal{G}' = \mathcal{A}.$$

"gaugings and ungaugings"

E.g. if $\mathcal{G} = n\mathrm{Vec}[G]$, $\mathcal{G}' = n\mathrm{Rep}(G)$.

• An operationally defined $(n+1)$D TQFT

is a (multi) fusion n-cat $\mathcal{A}$ of top'l ops

s.t. $Z(\mathcal{A}) \overset{\star}{=} n\mathrm{Vec}$. $\Leftrightarrow$ $\mathrm{Bim}(\mathcal{A}) = (n+1)\mathrm{Vec}$.

S-matrix $n > k > 0$

$$\Pi_k \mathcal{A} \times \Pi_{n-k} \mathcal{A} \to \mathbb{C}$$

$\star \iff$ nondeg $\forall k$. and when $k=0$,

statement is: if $\mathcal{A}$ fusion $\Leftrightarrow$ $\Pi_0 \mathcal{A} = *$.

e.g. $n+1 = 3$, lines pair w/ themselves

surfaces are all connected to id.

fusion case

$\Omega^n \mathcal{A} = \mathbb{C}$

$\Omega^{n-1} \mathcal{A} =$ lines

$\Omega^{n-2} \mathcal{A} =$ surfaces

$\vdots$

$\Omega^{n-k} \mathcal{A} = (n+1-k)$-monoidal k-cat.

$\vdots$

$\Omega \mathcal{A}$ braided $\otimes$ $n-1$ cat

$\mathcal{A}$ $\otimes$ n-cat

piece in top half.
if Tannakian
$\mathcal{A} /\!/ \Omega^{n-k} \mathcal{A}$.
kills $\pi_{\geq n-k}$.
$\Rightarrow$ also kills $\pi_{\leq k}$.
If $n+1 = 2k+2$,
maybe you can do whole top.

Stabilization hyp: If $n+1-k \geq k+2$,
then $\Omega^{n-k} \mathcal{A}$ is symmetric $\otimes$.

$\Omega^{n-k}\mathcal{A}$ $\qquad n+1-k \geq k+2,$

<u>Tannakian idea:</u> $\quad \Omega^{n-k}\mathcal{A} \xrightarrow{\text{fibre}} K\text{-Vec}$

iff $\quad \Omega^{n-k}\mathcal{A} \simeq K\mathrm{Rep}(G)$

now G is a K-gp.

$\overset{\shortparallel}{\mathrm{Aut}}(\text{fibre})$

$\sum$ $\quad \exists$ non-iso fibres.

$\mathrm{hom}_{\substack{\text{sym} \\ \otimes}}^{\circ}\big(2\mathrm{Vec}[\mathbb{Z}/_2],\ 2\mathrm{Vec}\big) = \mathbb{Z}/_2.$

as sym $\otimes$ cat

$\sim$ Tannakian duality over $\mathbb{K} \neq \overline{\mathbb{K}}$.