THE UNGAUGING SURGERY EXACT SEQUENCE

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"GENERALIZED SYMMETRIES IN QUANTUM FIELD THEORY:
HIGH ENERGY PHYSICS, CONDENSED MATTER, AND QUANTUM GRAVITY,"
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It's a pleasure to get to be here in beautiful Santa Barbara, and to have the opportunity to tell you about some of my work. I will tell you about a project with David Reutter. I'd like to say it is in progress, but right now "joint work" means that we basically know all the results, and haven't written enough, and we have each been distracted with various life events. Which is to say: when I say "Theorem" below, just know that the paper does not have ink on it. For those of you who have seen either of us speak before, this is part of our "deeper algebraic closure" aka "super duper vector spaces" aka "$\mathcal{W}$" project.

I only have an hour, so I won't try to tell you that whole story, but rather one interesting chapter of it.[1] Here is the context of the chapter. I am wandering down the street, and I come across an interesting TQFT. I'd like to take it apart into pieces, in such a way that I can reassemble the TQFT from the pieces, and also I'd like to do it in as canonical and straightforward a way as I can manage. If it is really canonical, then I would call this disassembly procedure a "classification." Perhaps "parameterization" is a better term, but "classification" is sexier. My goal in this hour is to tell you precisely how well (or poorly) this can be done.

1. Universal source: The Cobordism Hypothesis

Without specifying a bit more context, there is no chance I'd be able to say anything nontrivial about towards this goal. Indeed, the Cobordism Hypothesis teaches us that, if you are too open-minded about what counts as a "topological field theory," then you can think of any object of any (∞, n)-category (or at least any (∞, n)-category with enough duals) as some type of "topological field theory." And certainly there will not be a completely general procedure that disassembles arbitrary objects of arbitrary (∞, n)-categories. So from this perspective, the Cobordism Hypothesis is a very disappointing theorem if your goal is to classify TQFTs.

Actually, that's a bit too harsh of a perspective. The Cobordism Hypothesis does actually help, because it tells you what question matches what answer. Specifically, it tells you that the things which are objects of a higher category are *framed* TFTs, and the things that are morphisms are *framed* interfaces between them. Really I should say "coframed" rather than "framed". A *(co)framed (T)QFT* is a (T)QFT which is allowed to couple to a framing of spacetime — I like to think of that data as a sort of separation of scales between the different directions, completely breaking all microscopic Lorentz symmetry. Even if your (T)QFT happens not to couple to a

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[1]And, now that I've started writing these notes, I realize that they are way too long. So I will need to reorganize and cut some things before the talk. This is also a possible outline of a paper that might eventually be called "Deeper Galois Theory II," so it is not a waste of time to produce.

framing, when we talk about the space of (co)framed (T)QFTs, meaning the higher groupoid of such things up to isomorphism, then we implicitly mean that isomorphisms also are allowed to couple to a framing. A framing on a manifold is a lot of information, and it is hard to arrange: most manifolds do not admit framings. In turn, by allowing the (T)QFT to couple to a framing, it simplifies the amount of data needed to pin the TQFT — there are fewer spacetimes that it needs to make sense on! If you really want to study TQFTs — and their defects! — while demanding that they decouple from a framing and rather couple merely to an H-tangential structure for some H, then these are not the objects and morphisms of a category, but rather of what we are calling an "H-dagger category." But that is a story for another time: it is too new / in-the-works to talk about in a public setting like this.

So: my goal, physically, is to disassemble a framed TQFT. I still need more context to achieve this. Well, first, disassembly is dual to assembly. What tools should I insist on for assembly? I insist that my 0-dimensional TQFTs are plain $\mathbb{C}$-numbers, and that I am allowed to insert any k-dimensional TQFT as a decoupled k-dimensional operator into any $(n > k)$-dimensional ambient TQFT — indeed, k-dimensional TQFTs are nothing but the k-dimensional operators in the vacuum $(> k)$-dimensional TQFT. From the Cobordism Hypothesis perspective, "operator" just means "endomorphism," with some convention that you can work about relating the dimension of the operator to the dimension of the endomorphism. I insist that the 0-dimensional operators for a $\mathbb{C}$-vector space, and that higher-dimensional operators (in particular, TQFTs themselves) admit direct sums — that is a manifestation of some sort of "superposition" principle — and I insist that it is meaningful to condense a network of defects in my TQFT and thereby produce another TQFT, provided of course that the network is a (higher) "condensation algebra." These two insistences together already allow me to gauge finite higher group symmetries, for example. But they don't do enough to really specify which categories do or don't count as "categories of TQFTs."

2. Universal target: Anderson dual and category $\mathcal{W}$

The next insistence will make a big step towards that. Freed and Hopkins, in their work on invertible field theories, identified an important ansatz:

Freed–Hopkins Ansatz: A TQFT should be determined (up to isomorphism, which is unique up to *non-unique* higher isomorphism) by its partition function $\{\text{closed spacetimes}\} \to \{\text{numbers}\}$.

Actually, they only identified this for invertible TQFTs, and as stated it probably isn't quite true for noninvertible TQFTs. So instead I will take it as motivation. Now, the point of an ansatz is to guess it, work out what the answer must be in order for the ansatz to be fulfilled, and then prove that indeed the ansatz is fulfilled. Freed–Hopkins do this. Indeed, they knew the following nontrivial result:

Theorem [Anderson]: There is an E_∞ loop group, called $I^n_{\mathbb{C}^\times}$, which is determined (up to isomorphism, which is itself unique up to *non-unique* higher isomorphism) by the following formula. Suppose that E is any E_∞ loop group. Then

$$\pi_0 \hom(E, I^n_{\mathbb{C}^\times}) = \hom(\pi_n E, \mathbb{C}^\times),$$

where the LHS is hom of E_∞ loop groups, and the RHS is hom of abelian groups.

Thus, as long as "invertible nD TQFT" is set to mean "nD TQFT valued in $I^n_{\mathbb{C}^\times}$," then the Freed–Hopkins ansatz is satisfied. This is true no matter whether the cobordism hypothesis holds or not! I've written the theorem in terms of E_∞ loop groups, but it is formal from the characterization that the $I^n_{\mathbb{C}^\times}$s compile into a spectrum $I_{\mathbb{C}^\times}$.

What we did with David is to extend this to noninvertible land.

Theorem [JF–Reutter]: There is an E_∞-monoidal semisimple n-category with duals, called $\mathcal{W}^n$, which is determined (up to isomorphism, which is itself unique up to *non-unique* higher isomorphism) by the following formula. Suppose that $\mathcal{E}$ is any E_∞-monoidal rigid semisimple

n-category with duals. Then

$$\pi_0 \hom(\mathcal{E}, \mathcal{W}^n) = \hom(\pi_n \mathcal{E}, \mathbb{C}).$$

Semisimplicity builds in the ability to condense and to form direct sums. The LHS is hom of E_∞-monoidal n-categories. The LHS is hom of commutative algebras, where "$\pi_n \mathcal{E}$" means the commutative algebra of n-dimensional endomorphisms of the identity object in $\mathcal{E}$. I've written the theorem in terms of E_∞-monoidal categories, but it is formal from the characterization that the $\mathcal{W}^n$s compile into a categorical spectrum $\mathcal{W}$.

Comparing these statements implies the first of the following two criteria. A theorem of Deligne's implies the second.

(1) The group of invertible objects in $\mathcal{W}^n$ is isomorphic to $I^n_{\mathbb{C}^\times}$.
(2) The 1-category of line operators in $\mathcal{W}^n$ is isomorphic to $\mathrm{sVec}_{\mathbb{C}}$.

Theorem [JF–Reutter]: These criteria suffice to characterize $\mathcal{W}^n$.[2]

3. Ungauging

Access to category $\mathcal{W}$ gives an incredibly powerful disassembly/reassembly tool.

Theorem [JF–Reutter]: Suppose that $\mathcal{E}$ is a $\mathcal{W}^n$-linear E_∞-monoidal semisimple n-category with duals. Then the canonical map

$$\mathcal{E} \to \mathrm{maps}(\hom(\mathcal{E}, \mathcal{W}^n), \mathcal{W}^n)$$

is an isomorphism.[3] Conversely, suppose that $\mathcal{G}$ is a [4] n-groupoid. Then the canonical map

$$\mathcal{G} \to \hom(\mathrm{maps}(\mathcal{G}, \mathcal{W}^n), \mathcal{W}^n)$$

is an isomorphism.

Here "$\hom(\mathcal{E}, \mathcal{W}^n)$" means hom of $\mathcal{W}^n$-linear E_∞-monoidal n-categories, and is valued in n-groupoids; "maps" just means functors (from an n-groupoid to an n-category), and is valued in E_∞-monoidal n-categories.

Example: Any [5] semisimple commutative $\mathbb{C}$-linear algebra A is a product of copies of $\mathbb{C}$, one copy for each algebra map $A \to \mathbb{C}$ (=rank-1 projector in A).

Now, suppose I have an $n+1$D TQFT $\mathcal{Q}$. It has some n-category $\mathrm{End}(\mathcal{Q})$ of extendedoperators (or $n+1$-category, if you allow spacetime-filling operators), and my assumptions are that that n-category is $\mathcal{W}$-linear. Suppose I identify an E_∞-monoidal subcategory $\mathcal{E} \to \mathrm{End}(\mathcal{Q})$. The above theorem says I can write $\mathcal{E} = \mathrm{maps}(\mathcal{G}, \mathcal{W}^n)$. In other words: there is a "gauge groupoid" $\mathcal{G}$, and $\mathcal{E}$ is the category of "Wilson operators." Now I can ungauge this: there is a $\mathcal{G}$-parameterized family

$$\mathcal{Q}/\mathcal{E} : \mathcal{G} \to \{\mathrm{TQFTs}\}$$

such that if you re-gauge $\mathcal{G}$, you get $\mathcal{Q}$ back.

There are, of course, more general ungauging procedures by subcategories of operators that are not E_∞-monoidal. I won't need to use them, because of the following phenomenon:

[2] Actually, that's slightly false. They characterize the categorical spectrum $\mathcal{W}$ — the theorem is true if you work with multiple ns at the same time. For a fixed n, you could satisfy these criteria without being $\mathcal{W}^n$, but you will not be the loops of something that satisfies these criteria for $n+1$.

[3] Actually, that's slightly false, for set theory reasons. But it is true if $\mathcal{E}$ is *finite* semisimple.

[4] finite

[5] finite

4. Stabilization Hypothesis and middle-dimensional Chern–Simons fields

The *Stabilization Hypothesis* was formulated by Baez and Dolan in the same paper where they formulated the Cobordism Hypothesis. It is a now theorem[6] for weak n-categories, in particular for semisimple n-categories (and it is obviously not true for (∞, n)-categories). It asserts:

Stabilization Hypothesis: An E_p-monoidal weak q-category is automatically (canonically and uniquely) E_∞-monoidal as soon as $p > q + 1$.

This means you can do the following. Take your $n + 1$D TQFT $\mathcal{Q}$. Look inside $\mathrm{End}(\mathcal{Q})$ just at the operators of dimension $< \frac{n}{2}$. These operators form a q-category with $q < \frac{n}{2}$, and they are E_p-monoidal with $p = n + 1 - q > q$, and so symmetric monoidal. So this is a completely canonical E_∞-monoidal subalgebra of your full algebra, so you can ungauge it away.

So your $\mathcal{Q}$ is canonically the gauging of some family $\mathcal{G} \to \{\mathrm{TQFTs}\}$. In this family, $\mathcal{G}$ is a [7] q-groupoid, with q the largest integer strictly less than $\frac{n}{2}$. Moreover, because you ungauged them away, each member of this family if a TQFT with no operators of dimension $\leq q$.

A cute argument (namely: compute the Hilbert space of a torus in two different ways) then shows that all the operators of dimension $\geq n - q$ are condensates.

So if n is odd (meaning $n+1$ is even) then there are no operators at all, other than condensates of the vacuum. If n is even ($n+1$ is odd) then there might be some middle-dimensional operators. For example, that's what happens in WRT theories: there is a UMTC of middle-dimensional operators.

Lemma: A TQFT with no operators (other than condensates of the vacuum) is invertible; an invertible TQFT has no operators (other than condensates of the vacuum).

And recall that we already posited a classification of invertible operators in terms of $I_{\mathbb{C}^\times}$.

Corollary: If $n+1$ is even, then every $n+1$-dimensional TQFT is canonically a "finite homotopy theory" with gauge groupoid which is a $\frac{n-1}{2}$-groupoid.

5. Grouplike fusion

On the other hand:

Lemma [JF–Yu, essentially]: In a monoidal semisimple n-category with duals, if there are no operators of dimension ≤ 1, then all operators enjoy simple $\times$ simple $=$ simple fusion, and hence the condensation-equivalence classes of simple operators form a group.

In the case under consideration, as soon as $n + 1 > 3$, then the ungauging gets rid of all the (≤ 1)-dimensional operators. Suppose that we do have some middle-dimensional operators left after ungauging — meaning $n + 1$ is even, so that the middle-dimensional operators live in dimension $\frac{n}{2}$. Then, from the lemma, these middle-dimensional operators form an abelian group A, and the whole algebra of operators is classified by this group A and a class

$$\omega \in \tilde{I}^{2k}_{\mathbb{C}^\times}(K(A, k)) := \pi_0 \operatorname{maps}_*(K(A, k), I^{2k}_{\mathbb{C}^\times}),$$

where $k = \frac{n}{2} + 1 = \frac{n+2}{2}$.

Gut check: braided fusion categories with simple $\times$ simple $=$ simple fusion are classified by an abelian group A and a class $\omega \in H^4_{\mathbb{C}^\times}(K(A, 2))$; the switch from H to I comes when you work with super MTCs rather than bosonic MTCs. By "maps_*" I mean reduced cohomology — I want cohomology classes trivialized at the basepoint in $K(A, k)$ — that's what the tilde is doing on the I. Note that $K(A, k)$ is a spectrum, but I do not mean spectrum maps: I mean the cohomology of the Eilenberg–Mac Lane space $K(A, k)$.

Which classes ω actually come from true honest (absolute) TQFTs? This takes a couple steps to answer. I will need to use some of the "differential calculus" aka "Taylor theory" for Eilenberg–Mac

[6] nLab says that the main idea is due to Simpson in 1998, with a full proof first sketched by Joyal 2008 and first written completely by Lurie in 2017, and in a neat time-traveling trick Gepner and Haugseng then gave an improved proof in 2013.

[7] finite, but this is getting boring to write, so I will stop

Lane cohomology. Here's what you can do. By Hurewicz, $\mathrm{maps}(S^k \times S^k, K(A,k)) = A \times A$. So pick $(a,b) \in A \times A$, and restrict your $\omega \in I^{2k}_{\mathbb{C}^\times}(K(A,k))$ to a class in $I^{2k}_{\mathbb{C}^\times}(S^k \times S^k)$, and now integrate this over the big cell to get just a plain $\mathbb{C}^\times$-number, mayby I'll call it $\varpi(a,b)$. This gives a function $\varpi : A \times A \to \mathbb{C}^\times$. It is not too hard to show that ϖ is a bilinear pairing, and is either symmetric or skew-symmetric according to the parity of k.

Proposition [JF–Reutter, with a hint from Hopkins]: This map

$$\omega \mapsto \varpi : \tilde{I}^{2k}_{\mathbb{C}^\times}(K(A,k)) \to \{(\text{skew})^k\text{-symmetric bilinear forms on } A\}$$

is an isomorphism of sets, meaning an isomorphism on π_0.

Actually, we can do better. $K(A,k)$ is an (higher) abelian group. So it makes sense to talk about linear functions, quadratic functions, etc. Let's write $\mathcal{P}_{\leq d} I^j_{\mathbb{C}^\times}(K(A,k))$ for the maps $K(A,k) \to I^j_{\mathbb{C}^\times}$ of polynomial degree $\leq d$.

Proposition: The map

$$\mathcal{P}_{\leq 2}\tilde{I}^{2k}_{\mathbb{C}^\times}(K(A,k)) \to \tilde{I}^{2k}_{\mathbb{C}^\times}(K(A,k))$$

is an isomorphism of spaces.

Thus, since it is a map of infinite loop groups, it is in fact an isomorphism of infinite loop groups. Warning: It is not an isomorphism of spectra — it is not an isomorphism in $\pi_{\ll 0}$.

A necessary condition for (A,ω) to come from a $(2k-1)$-dimensional TQFT is that the bilinear form ϖ is nondegenerate. Let's say that ω is "nondegenerate" when this happens. This is a modularity / remote detectibility condition. Suppose that it is nondegenerate. Then you can build a $2k$-dimensional finite path integral TQFT. Since ω is quadratic, this $2k$-dimensional TQFT is a "Gauss sum" theory, and we have

Proposition [Gauss, essentially]: Suppose that $\mathcal{A}$ is any π-finite spectrum, concentrated in degrees $1,\dots,m-1$, and $\omega \in \mathcal{P}_{\leq 2} I^m_{\mathbb{C}^\times}(\mathcal{A})$ is a quadratic $I_{\mathbb{C}^\times}$-valued function. The m-dimensional Gauss-sum TQFT built from $\mathcal{A},\omega$ is invertible iff ω is nondegenerate.

Finally, if you have A and a nondegenerate $\omega \in \tilde{I}^{2k}_{\mathbb{C}^\times}(K(A,k))$, then this invertible $2k$-dimensional TFT comes with a Dirichlet boundary condition (that's what the tilde does), and the boundary operators are precisely this fusion higher category built from A,ω. So for that category to come from an honest TQFT is simply for the invertible $2k$-dimensional bulk theory to be trivial(ized).

6. L theory and the exact sequence

To summarize so far, given a TQFT of spacetime dimension > 3, we can canonically condense all the low-dimensional operators and end up with a Gauss sum theory (A,ω) with the property that its universal bulk aka anomaly is trivial(ized). What if you have an interface between such theories? Well, if the defect itself has spacetime dimension > 3 — if the theories have spacetime dimension ≥ 5 — then you can do the same thing on the defect. In other words, if $n \geq 5$, we're finding a sequence, exact in the middle, looking like:

$$\begin{aligned}&\{\text{simple } n\text{D TQFTs}\}/\{\text{simple interfaces}\}\\ &\qquad \to \{\text{Gauss-sum theories}\}/\{\text{Gauss-sum interfaces}\}\\ &\qquad\qquad \to \{\text{invertible } (n+1)\text{D TQFTs}\}/\{\text{invertible interfaces}\}.\end{aligned}$$

And the kernel of the first map is precisely the invertible theories, since a TQFT is invertible iff it has no operators (other than condensates of the vacuum). So actually you have an exact triangle aka LES.

We have names for two of the three corners in this triangle. First, {Invertible nD TQFTs} is called $\pi_{-n} I_{\mathbb{C}^\times} = I^n_{\mathbb{C}^\times}(\mathrm{pt})$. It is the Pontrjagin dual group to $\pi_n \mathbb{S} = \pi_n \mathrm{Mfr}$, the nth stable homotopy group of spheres.

Second, there is a spectrum built as follows. A degree-m cocycle is a π-finite spectrum $\mathcal{A}$ equipped with a degree-m nondegenerate quadratic function $\omega \in \mathcal{P}_{\leq 2}\tilde{I}^m_{\mathbb{C}^\times}(\mathcal{A})$. Two cocycles are cohomologous when they can be related by a Lagrangian correspondence. Here is perhaps a better description. Cochains (of degree m) for this cohomology theory are pairs $(\mathcal{A} \in \mathrm{Sp}_{\pi f}, \omega \in \mathcal{P}_{\leq 2}\tilde{I}^m_{\mathbb{C}^\times}(\mathcal{A}))$. Recall above that the quadratic function ω induces a bilinear pairing ϖ, equivalently a map $\varpi : \mathcal{A} \to \mathcal{A}^\vee$ (=linear functions from $\mathcal{A}$ to $I_{\mathbb{C}^\times}$), and ω is nondegenerate iff this map is an iso. When it's not an iso, its (co)fibre comes with its own quadratic function, always nondegenerate, of degree $m+1$. So we set $\mathrm{d}(\mathcal{A}, \omega) = (\mathrm{fib}(\varpi))$.

To build this spectrum, it didn't really matter that I required $\mathcal{A}$ to be π-finite, or that I was using $I_{\mathbb{C}^\times}$-valued quadratic functions. What mattered was that I had some [stable $(\infty, 1)$-]category of "abelian groups" — in my case this category is $\mathrm{Sp}_{\pi f}$ — and some notion of "quadratic function" — in my case, $\mathcal{P}_{\leq 2}\tilde{I}_{\mathbb{C}^\times}$ — with some formal properties allowing you to talk about the induced bilinear pairing, and to ask whether that pairing is nondegenerate. These input data are called "stable category equipped with a nondegenerate quadratic functor."[8] If you input $(\mathcal{C}, Q)$, the output spectrum is called the *L-theory of* $(\mathcal{C}, Q)$, denoted $L(\mathcal{C}, Q)$.

Main Theorem: There is a long exact sequence, starting at $n = 5$, of the form

$$I^n_{\mathbb{C}^\times} \longrightarrow \frac{\{\text{simple } n\text{D TQFTs}\}}{\text{interfaces}} \longrightarrow L^{n+1}(\mathrm{Sp}_{\pi f}, \mathcal{P}_{\leq 2}\tilde{I}_{\mathbb{C}^\times}) \longrightarrow I^{n+1}_{\mathbb{C}^\times} \to \dots$$

Actually calculating the groups $I^n_{\mathbb{C}^\times}$, equivalently the homotopy groups of spheres, is notoriously difficult. On the other hand, the groups $L^n(\mathrm{Sp}_{\pi f}, \mathcal{P}_{\leq 2}\tilde{I}_{\mathbb{C}^\times})$ are easy:

Proposition [Witt, essentially]:

$$L^n(\mathrm{Sp}_{\pi f}, \mathcal{P}_{\leq 2}\tilde{I}_{\mathbb{C}^\times}) = \begin{cases} 0, & n \equiv 1 \pmod 2, \\ \mathbb{Z}/2, & n \equiv 2 \pmod 4, \\ W, & n \equiv 0 \pmod 4, \end{cases}$$

where W is the Witt group of finite abelian groups with a nondegenerate symmetric bilinear form:

$$W = \bigoplus_{p \text{ prime}} \begin{cases} \mathbb{Z}/2, & p = 2, \\ (\mathbb{Z}/2)^2, & p \equiv 1 \pmod 4, \\ \mathbb{Z}/4, & p \equiv -1 \pmod 4. \end{cases}$$

And the map $L^n \to I^n_{\mathbb{C}^\times}$ is known, because it's just a question of evaluating Gauss sums for manifolds. What you find out is:

Proposition When $n \equiv 2 \pmod 4$, the map $\mathbb{Z}/2 = L^n \to I^n_{\mathbb{C}^\times}$ selects the Arf–Kervaire invariant. When $n \equiv 0 \pmod 4$, the map $W = L^n \to I^n_{\mathbb{C}^\times}$ is zero.

Finally, a combination of hard theorems of Browder, Mahowald–Tangora, Barratt–Jones–Mahowald, Hill–Hopkins–Ravenel, and Lin–Wang–Xu tells you that:

Theorem (hard, due to ibid): The Arf–Kervaire invariant is nonzero — equivalently, the map $L^n \to I^n_{\mathbb{C}^\times}$ is nonzero — if and only if $n = 2$, 6, 14, 30, 62, or 126.

In other words, what I'm saying is the following. Suppose that $\mathcal{I}$ is a nontrivial invertible TQFT. Then, unless $\mathcal{I}$ is one of these six Arf–Kervaire invariants, $\mathcal{I}$ has no topological defects to the vacuum. In other words, any boundary condition of $\mathcal{I}$ is necessarily nontopological aka gapless.

So this is an "enforced gaplessness" result: if you have a propertyl-anomalous QFT, and if the anomalous is not one of these six Arf–Kervaire invariants, then your QFT is gapless.

[8]Note the nice example of the cosmic principle. Stable categories are the $(\infty, 1)$-categorification of abelian groups / spectra. To talk about things with a "quadratic function," you need your ambient category to come with a "quadratic functor." To talk about "nondegenerate quadratic functions," you need your quadratic functor to be nondegenerate.