# On Atoms, Mountains, and Rain

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**Abstract**

This talk won't include very many *facts*, but it will include many *almost facts*, aka "lies". A few lies we will tell: rocks are made of rock atoms, liquid water is a perfect cubic crystal lattice, and $1 = 2$. Using these and similar "facts", we will derive from first principles the radius of an atom, the height of a mountain, and the volume of a raindrop. Doing so honestly, even if we knew all the fundamental equations of the universe, would be impossible; lying makes everything work out nicely.

## 1 Introduction

Before I begin, I should give credit where credit is due. Everything I say is from:

> P. Goldreich, S. Mahajan, and S. Phinney, *Order-of-Magnitude Physics: Understanding the World with Dimensional Analysis, Educated Guesswork, and White Lies*, 1999. Available at `http://www.inference.phy.cam.ac.uk/sanjoy/oom/`.

What's available online is only the first half of a textbook, which as far as I know has not been published. A few more chapters are available as part of Sanjoy Mahajan's related class at MIT. The material also appeared in Sanjoy Mahajan's PhD thesis. We quote his motto from the book:

"When the going gets tough, lower your standards."

### 1.1 Units

In physics, there are pure "unitless" numbers, but there are also "unitful" quantities: distances, lengths of time, energies, and the like. One cannot add unitful numbers unless their units match — they live in different vector spaces — but one can multiply them; the result lives in the tensor product of the vector spaces. Moreover, unitful numbers are not in the domains of the mathematicians' functions: it does not make sense to multiply a number by itself "2 inches"-many times, for example. We have a rather philosophical axiom:

**Axiom 1.** *Physical theories should never involve unitful constants.*

All this really means is that if your theory includes a unitful constant — the speed of light, for example, or Plank's constant — then you should replace it with a letter — $c$ or $\hbar$ — and use it as a variable. There is a bit more depth to this, however. For a physicist, all functions are smooth, and by treating $c$ and $\hbar$ and the like as variables, we can bring to bear all the mathematical theorems about the behavior of functions. At the end of the day you may specialize to the measured values of your physical quantities, but the logic you use to derive your formulas should not depend on quantities taking specific values.

We now "tropicalize" our unitful quantities, considering only the units on that quantity. For instance, a length we might write as

$$[l] = \mathrm{cm}$$

for "centimeters", and an energy as

$$[E] = \mathrm{g\,cm^2/s^2}$$

We've used "cgs" units because they make electromagnetism easier, by normalizing $\epsilon_0$ and $\mu_0$ to 1. In particular, in cgs, the units of electric charge are:

$$[e^2] = \frac{\mathrm{g\,cm^3}}{\mathrm{s^2}}$$

Here and throughout, $e$ is the charge of the electron — we will never need Euler's base of the natural exponential — and we will generally work with $e^2$ rather than $e$ because we don't particularly like talking about square roots of grams, and because we can never remember whether the charge of an electron is supposed to be "positive" or "negative".

In any case, upon such tropicalization, the only effect of multiplication of unitful quantities is to add the exponents of the corresponding units. In particular, each vector space worth of quantities corresponds to a point in a single vector in "unit-space". For example, if the coordinates of unit space begin (mass, distance, time, ...), then the above amounts are:

$$[l] = (0, 1, 0, \dots),\ [E] = (1, 2, -2, \dots),\ \text{and } [e^2] = (1, 3, -2, \dots)$$

The following is a basic piece of linear algebra:

**Theorem** (Buckingham Pi)**.** *If we are given $n$ variables taking values in combinations of $m$ basic independent units, then we should expect to be able to form $n - m$ "dimensionless groups".*

Given $l, E, e^2$ above, the only dimensionless group is $El/e^2$. By Axiom 1, any physical theory that depended only on the quantities $l$, $E$, and $e^2$ must be of the form $f(El/e^2) = 0$ for some function $f$. But in physics all functions are invertible, and so this physical theory actually takes the form $El/e^2 = \text{constant}$.

To illustrate this "unit analysis" further, we describe the gravity pendulum. Say we're interested in computing the period $T$ of a pendulum with length $l$ and mass $m$, in an environment with ambient gravitational acceleration $g$, in terms of the angular amplitude $\theta$. We have:

$$[T] = \mathrm{s},\ [l] = \mathrm{cm},\ [m] = \mathrm{g},\ [g] = \mathrm{cm/s^2},\ [\theta] = 1$$

The dimensionless groups are $gT^2/l$ and $\theta$, and so the physics of a pendulum consists of precisely a relationship between these two quantities: $gT^2/l = F(\theta)$. Solving for the period, we have:

$$T = f(\theta)\sqrt{\frac{l}{g}}$$

where $f(\theta) = \sqrt{F(\theta)}$. Honest physics and differential equations (or, in my case, looking up the answer on Wikipedia) confirms this and gives the asymptotics of $f(\theta)$:

$$f(\theta) = 2\pi\left(1 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4 + \dots\right)$$

But without any true physics, we can see that the mass of the bob on the end of a pendulum (with an otherwise massless string; if the string had mass, we'd have another unitful quantity and hence another dimensionless group) does not affect the period of oscillation.

## 1.2 Approximate Mathematics

Up to now, the only approximation we have used is to assert that all functions are invertible, which is at least locally true. We now begin to approximate in earnest. We introduce the following "mathematical" axiom:

**Axiom 2.** *All unitless numbers are roughly unity, zero, or infinity. Zero and infinity don't really count as numbers.*

**Theorem.** *The minimum value of $f(x) + g(x)$ occurs when $f(x) = g(x)$.*

*Proof.* All functions are monomials, so $f(x) = Ax^n$ and $g(x) = Bx^m$. If $n$ and $m$ have the same sign, then $f(x) + g(x)$ is monotonic. So we assume without loss of generality that $m$ is negative and $n$ is positive. We will switch $m \mapsto -m$. The minimum occurs at the zero of $\frac{d}{dx}\big[f(x) + g(x)\big] = \frac{d}{dx}\big[Ax^n + Bx^{-m}\big] = nAx^{n-1} - mBx^{-m-1}$. I.e. it occurs when $nAx^{n-1} = mBx^{-m-1}$, i.e. $n\,f(x) = m\,g(x)$. But $n/m = 1$. □

The most important instance of this theorem is:

**Corollary** (Virial Theorem)**.** *In any physical system, total kinetic energy equals total potential energy.*

*Proof.* By the laws of thermodynamics, physical systems tend towards lowest-energy states. In essence, if a system is not in a lowest-energy state, it will have a tendency to lose energy to its environment. Thus, the probability of every coming across a system not in a lowest-energy state is astronomically small, because such states don't stick around. Since total energy is exactly kinetic energy plus potential energy, the result follows from the above theorem. □

We remark that not every function is a monomial, but almost all are. Let $f(x)$ be a function. If $f(x)$ is smooth near 0, then we can expand $f$ in Taylor series, and only the first few terms survive: $f(x) = f(0) + \sum_{n=1}^{5} A_n x^n$. So $f'(x) = \sum_{n=1}^{5} n A_n x^{n-1} = \big(f(x) - f(0)\big)/x$, since $n = 1$ for $n \leq 5$. But functions should not have constant terms, at least not unit-ful constant terms. So $f(0) = 0$ for unit-ful functions like Energy, and so generally we can "cancel the $d$s":

$$\frac{df}{dx} = \frac{f}{x}$$

If on the other hand $f(x)$ is smooth near $\infty$ and has a pole near 0, then we can make a similar expansion in $x^{-1}$, and now canceling the $d$s is correct up to a minus sign:

$$\frac{df}{dx} = -\frac{f}{x}$$

Of course, signs are notoriously hard to do correctly, because there is no good sign convention. On the other hand, physical intuition is almost always good enough to determine the sign of the final answer, so it usually suffices to compute only the magnitude of a quantity correctly. Another motto is extremely important to keep in mind:

The difference between a good mathematician and a bad one
is the parity of the number of sign errors.

## 2 Atoms

Following Mahajan's book, we quote the great Richard Feynman (R. Feynman, R.B. Leighton, and M. Sands. *The Feynman Lectures on Physics*, vol. I. Addison-Wesley, Reading, MA, 1963, pp. 1–2):

> If, in some cataclysm, all of scientic knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the atomic *fact*, or whatever you wish to call it) that *all things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another.* In that one sentence, you will see, there is an enormous amount of information about the world. . . .

As mathematicians, we formalize this into a (physical) "axiom":

**Axiom 3.** *All matter is made of atoms. For example, water is made of water atoms, and rock is made of rock atoms.*

Let us compute the important quantities concerning atoms. An atom, as you know, consists of a dense, heavy nucleus surrounded by a cloud of light electrons. The hydrogen atom is simplest: the nucleus is a single proton, and there is a single electron.

## 2.1 Size of an atom

What quantities might affect the radius $r$ of a hydrogen atom? The masses $m_e$ and $m_p$ of the electron and the proton, certainly, and the square charge $e^2$, but so far the only dimensionless group is $m_e/m_p \approx 0$. Since this ratio is roughly zero ($m_p/m_e = 2000$ or so), one of the quantities is almost certain to drop out; we will argue in a moment that $m_p$ does not effect the size of an atom. But we need another variable. Well, what keeps an electron cloud from collapsing? Quantum Mechanics must certainly enter the picture, and so we introduce the variable $\hbar$. Checking the units, we have a unique dimensionless group $r m_e e^2/\hbar^2$, which must be a constant by Axiom 1, and this constant must be 1 by Axiom 2. Thus:

$$r = \frac{\hbar^2}{m_e e^2} = a_0$$

This quantity is called the "Bohr radius" of the hydrogen atom.

Let us justify this with some physics. By the Virial Theorem, we should compute the kinetic and potential energies of a system. Now, if the atom is at rest, it has total momentum 0, and so the momenta of the nucleus and of the electron are equal and opposite. But kinetic energy is

$$\text{KE} = p^2/m$$

and so the kinetic energy of the proton is astronomically smaller than the kinetic energy of the electron. This is why the mass of the proton should not matter for determining the radius of the atom. Thus, the total kinetic energy is $\text{KE} = p^2/m_e$. On the other hand, the total potential energy is determined by Coloumb's Law: $\text{PE} = e^2/r$. Lastly, Heisenberg's uncertainty principle gives $pr = \hbar$, since $p = \Delta p$ and $r = \Delta x$, at least up to sign. Thus $p = \hbar/r$, and so we solve:

$$\frac{(\hbar/r)^2}{m_e} = \frac{e^2}{r}$$

This yields the same result: $r = a_0$.

More generally, an atom may have $n$ protons and $n$ electrons. Imagine starting with $n$ protons, and adding the electrons one-by-one. Certainly the first electron will fall much closer to the nucleus; indeed, from its perspective, the potential energy is $ne^2/r$, and so it will live at height $a_0/n$. But from the perspective of the second electron, the central charge is partially shielded; it only sees $n-1$ protons. And the last electron sees just one, so we expect it to live at size $a_0$.

To check the numbers, we quote some handy facts. $\hbar c = 2000\,\text{eV}\,\text{Å}$, where "eV" means "electron-volt", a unit of energy, and "Å" means "angstrom", or $10^{-10}$ meters. $m_e c^2 = 500\,000\,eV$. And $e^2/\hbar c = 0.01$ is unitless; this is called the "fine structure constant". The fact that this (and also $m_e/m_p$) violate Axiom 2 is one of the great mysteries of physics, and a leading motivation for research like String Theory.

Anyway, we see that:

$$a_0 = \frac{\hbar^2}{m_e e^2} = \frac{\hbar c}{m_e c^2}\frac{\hbar c}{e^2} = \frac{2000\,\text{eV}\,\text{Å}}{500\,000\,\text{eV}}\frac{1}{0.01} = 0.5\,\text{Å}$$

In fact, the true radius of Hydrogen and Helium atoms is about 1 Å, Lithium through Neon are about 2 Å, and all the rest are about 3 Å, so we're not far off. $6 \approx 1$, in confirmation of Axiom 2.

### 2.2 Ionization Energy, Bond Breaking

We compute now the first ionization energy of an atom, because we will need it for later. This is the energy required to move an electron against a Coloumb force from its atomic radius out to infinity, equal exactly to $e^2/r$. When $r = a_0 \approx 0.5\,\mathring{A}$, this is:

$$\frac{e^2}{r} = \frac{m_e e^4}{\hbar^2} = m_e c^2 \left(\frac{e^2}{\hbar c}\right)^2 = 50\,\mathrm{eV}$$

Of course, since the true radius is a bit larger than $r$, this is actually a bit of an over estimate; ionization energies are closer to 10 eV. True values range from 4 eV at the lowest end (Francium), to 24 eV (Helium) at the upper end.

To break a bond requires almost as much energy as ionization. Indeed, an atomic bond between two atoms is as if each has its outermost electron shared with the other atom. Breaking this bond is like ionizing half an electron from each.

The volume of an atom is just $r^3$, up to some small constants, so ionization energy per unit volume is $e^2/r^4$. Well, $e^2/a_0^4 = (50\,\mathrm{eV})/(0.5\,\text{Å})^3 = 10^{14}$ pascals. However, now the fact that our theoretical radius and measured radius are off by a factor of roughly 6 really matters: $6^4 = 1000$, so the ionization energy per volume (of solid or liquid) is roughly $10^{11}$ pascals.

# 3 Mountains

We now have the tools necessary to compute the height of a mountain. Of course, mountains range in sizes, so really we will compute the height of the tallest mountain. There are too many parameters involved to simply use dimensional analysis. Rather, we need some physics.

Mountains are built by piling rock up, perhaps by volcanoes, and are shrunk by erosion. But the tallest mountains at any given time haven't had a chance to be worn down by erosion — that's a slow process, whereas volcanoes are fact. So what keeps them from growing infinitely tall? Gravity pulls down on the giant pile of rock. And eventually the mountain gets so tall that the force of compression from all that weight is enough to start to melt the rock at the bottom. Good thing this didn't happen in Dr. Seuss's *Yertle the Turtle*.

Gravity pushes straight down, so the only part of the mountain that matters for determining its height is the central column. Let's consider a column of rock with base $b$ ($[b] = \mathrm{cm}^2$), height $h$, and density $\rho$. If a volume $V = b\eta$ of the rock at the bottom of the column were to melt, it would allow the whole column to shift down by a distance $\eta$. Thus, the gravitational potential energy released would be $\rho b h\, g\, \eta = \rho V g h$.

We've computed already the energy required to fully ionize every outer electron in a volume $V$ of rock. Does melting rock require this energy? We should model rock as a cubic crystal, each atom bonded to six neighbors, and each bond shared between two atoms. So each atom has three bonds of its own. So to release every atom, then, requires breaking three bonds per atom. However, to melt the rock perhaps requires breaking only one one-thousandth of the bonds. Thus, we expect the energy required to melt a volume $V$ of rock to be roughly $0.003 \times V \times 10^{11}$ pascals, since $10^{11}$ pascals was the energy per unit volume required to break one bond per atom.

We set these two energies equal, cancel the $V$s, and solve for $h$. We have:

$$h = \frac{1}{\rho g} \times 3 \times 10^8 \text{ pascals}$$

Using $\rho = 3\,\text{g/cm}^3$ and $g = 10\,\text{m/s}^2$, we have $h = 10$ kilometers. In fact, Mt. Everest is almost exactly this height: 8.8 kilometers.

Ok, so I fixed the numbers a bit when writing this talk. I actually have no idea exactly how many bonds are required to melt rock; I did not distinguish between rock and diamond, and if you guess that it takes one one-hundredth of the bonds to break before the rock melts, you get an answer of 100 km. More generally, perhaps the mountain need not actually *melt* the rock underneath, so much as break enough bonds that the bottom rock at least stops supporting the mountain. But I did prove:

**Theorem.** *The height of the tallest mountain on the surface of a planet is inversely proportional to the strength of gravity.*

This, in fact, holds up very well. Mars, for example, has one third the gravity Earth has, and sure enough Olympus Mons is almost precisely 27 kilometers tall.

Also, it is an easy exercise to check that the gravity on the surface of a rocky planet is proportional to the radius of the planet, and so we have shown:

**Corollary.** *Let $R$ be the radius of a rocky planet, and $h$ the height of its tallest mountain. The unitful product $Rh$ does not depend on the planet (i.e. it is a constant).*

So how large can an asteroid be before it must be roughly spherical? A non-spherical asteroid is essentially a sphere with a same-sized mountain stuck on the side. Since on Earth, $R = 6000$ km, and $h = 9$ km, we see that $Rh = 50\,000\,\text{km}^2$. If $R = h$ on our nonspherical asteroid, we would must have $R = h = \sqrt{50\,000\,\text{km}^2} = 200\,\text{km}$. This is just about what is observed in our solar system.

# 4 Raindrops

We have described the heights of mountains. We turn now to something much smaller: raindrops.

## 4.1 Size of a raindrop

What makes a raindrop the size it is? Well, what makes a raindrop try to be large? You may have heard of "surface tension", which is the tendency of water to try to clump with other water. What makes a raindrop try to be small? You may have heard of "wind resistance"; the force of the air the raindrop falls through tries to break up the droplet. We will address each of these in turn.

We can understand surface tension in terms of the atomic hypothesis:

**Theorem.** *A water atom would prefer to be surrounded by other water atoms, rather than having one side exposed.*

The proof requires only the discussion of bond-breaking from previously.

*Proof.* Water, like all other materials, forms a cubic lattice: each water atom in the interior of a droplet is bonded to six neighbors. However, a water atom on the surface of a droplet has only five neighbors. Thus, to create more atom worth of surface area requires breaking one bond. This creates $\pi r^2$-much surface area, where $r$ is the radius of an atom. Then the energy cost per unit surface area to a droplet of water is:

$$\frac{e^2}{r}\frac{1}{\pi r^2} = \frac{e^2}{\pi r^3} = \frac{\hbar c \times (e^2/\hbar c)}{\pi\, r^3} = \frac{2000\,\mathrm{eV\,\AA} \times 0.01}{\pi\,(3\,\mathrm{\AA})^3} = 0.25\,\mathrm{eV/\AA^2} = 4000\,\mathrm{erg/cm^2} = 4000\,\mathrm{g/s^2}$$

This is actually too large by a factor of about 50. This is because water is actually molecules, not atoms. The molecules are bonded much more loosely to each other, lowering the energy required to break a bond, and are larger (say twice the radius), lowering the number of bonds required to create a given area. Anyway, so let's call the above number $\gamma$, for "surface tension".

Let us now calculate the drag force caused by the air resistance. In general, calculating drag force is very difficult: it depends on the speed of the object and on the geometry. But we can skip this difficulty by assuming that the raindrop is traveling at terminal velocity. Then the drag force, which pushes up, must equal the gravitational force pulling down. This gravitational force is a constant $\frac{4}{3}\pi R^3 \rho g$, where $\rho$ is the density $\rho = 1\,\mathrm{g/cm^3}$ of water, and $g$ is the constant acceleration due to gravity $g = 1000\,\mathrm{cm/s^2}$. So the drag force at terminal velocity on a raindrop is precisely $\frac{4}{3}\pi R^3 \rho g \approx R^3 \rho g$.

We could convert this into an energy cost of having too large of a raindrop. But instead we will convert the surface tension energy into a force. The surface tension $\gamma$ is energy per unit area, but this is also force per unit length. So the surface-tension force is just $\gamma R$, since $R$ is the only length dimension. We set these two forces equal and solve for $R$:

$$R^3 \rho g = \gamma R \text{ and so } R = \sqrt{\gamma/\rho g}$$

Another way to say this is: we've figured out that the size $R$ of a raindrop should be determined by the dimensionful parameters $\rho$, $g$, and $\gamma$, with units $[\rho] = \mathrm{g/cm^3}$, $[g] = \mathrm{cm/s^2}$, and $[\gamma] = \mathrm{g/s^2}$. The only dimensionless group is $R^2 \rho g/\gamma$, and so this must be a constant by Axiom 1, which must be roughly 1 by Axiom 2.

We remark that $\gamma$ is inside the square root, so any error isn't too bad. Using our calculated $\gamma = 4000\,\mathrm{g/s^2}$, we have:

$$R = \sqrt{\frac{4000\,\mathrm{g\,s^{-2}}}{1\,\mathrm{g\,cm^{-3}} \times 1000\,\mathrm{cm\,s^{-2}}}} = 2\,\mathrm{cm}$$

The measured $\gamma$ is fifty times smaller, and $\sqrt{50} = 7$; the actual size of a raindrop really is just about a quarter of a centimeter.

## 4.2 Raindrop speed

Lastly, let's compute the terminal velocity. The terminal velocity $v$ of an object depends on its radius $R$, the force of gravity $g$, the densities $\rho_{\mathrm{air}}$ of the fluid the object is moving through and $\rho_{\mathrm{water}}$ of the object, and the viscosity $\nu$ of the fluid. But in the high-speed limit, one can prove that the viscosity does not matter. Indeed, a raindrop goes fast enough that the air is simply shoved out of the way; only in the slow limit does momentum dissipate viscosity. In any case, the dimensionless groups are $\rho_{\text{water}}/\rho_{\text{air}}$ and $v^2/gR$; thus:

$$\frac{v^2}{gR} = f\left(\frac{\rho_{\text{water}}}{\rho_{\text{air}}}\right)$$

for some function $f$. Certainly $f(x)$ must be $x^n$ for some positive $n$, since the terminal velocity increases with the density of the object. We guess the simplest such function — $f(x) = x$ — and more honest physics, which we will not describe here, verifies that this is the right guess.

We quote the fact that $\rho_{\text{water}}/\rho_{\text{air}} = 1000$. Thus:

$$v = \sqrt{gR\frac{\rho_{\text{water}}}{\rho_{\text{air}}}} = \sqrt{1000\,\text{cm}\,\text{s}^{-2} \times 0.25\,\text{cm} \times 1000} = 500\,\text{cm/s} \approx 10\,\text{mph}$$

This is very reasonable: when you drive slowly, rain falls at a 45° angle. □

# 5 Conclusion

In conclusion, physics is cool, and much easier than math.