Pseudounitary slightly degenerate braided fusion categories admit minimal modular extensions

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Goal for the talk:

Outline proof of the title.

These slides available: categorified.net/MPPMslides.pdf

Definition Recollection:

- The Müger centre $Z_{(2)}\mathcal{B}$ of a braided fusion cat $(\mathcal{B}, \otimes, \beta)$ is the full subcat on those x s.t. $\beta_{x,y} \circ \beta_{y,x} = id \;\; \forall y \in \mathcal{B}$. It is a symmetric fusion category.

- $\mathcal{B}$ is slightly degenerate if $Z_{(2)}\mathcal{B} \simeq SVec$, and nondegenerate if $Z_{(2)}\mathcal{B} \simeq Vec$.

- A minimal modular extension (MME) of $\mathcal{B}$ is an inclusion $\mathcal{B} \subseteq \mathcal{C}$ where $\mathcal{C}$ is nondegenerate and:

minimality condition

$$\{x \in \mathcal{C} \text{ s.t. } \beta_{x,y} \circ \beta_{y,x} = id \;\; \forall y \in \mathcal{B}\} = Z_{(2)}\mathcal{B}$$

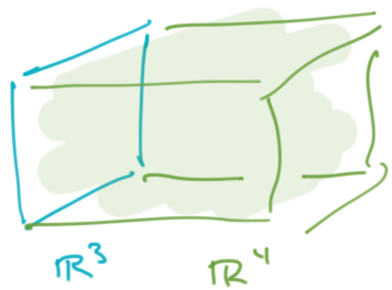
Warning: $\exists \mathcal{B}$ w/ $Z_{(2)}\mathcal{B} \simeq Vec[\mathbb{Z}_2^2]$ s.t. no MME exists.

How to think of $\mathbb{B}$?

↪ skein theory in 3D (2+1D). ↪ 4D (3+1D) Crane–Yetter–Walker–Wang TQFT (framed, for now).

These are related: the 4D CYWW model has a boundary condition.

$\mathbb{B}$ = line operators on the boundary



$Z_{(2)}\mathbb{B}$ = line operators in the bulk.

There are also higher-dim operators (surfaces, branes, ...). The n-dim operators form an (E_{3-n}-monoidal) fusion n-category.

On the boundary, the surfaces are the fusion 2-category

$$\Sigma\mathbb{B} := \mathrm{Mod}(\mathbb{B}) = \{\text{algebras, bimods in } \mathbb{B}\}$$

$\boxed{\text{Determined by } \mathbb{B}}$

In the bulk, the surfaces form $Z(\Sigma\mathbb{B})$

↑ Drinfeld centre of fusion 2-cat

$\boxed{\text{N.B: This br. 2-cat determines the } \otimes \text{ 3-cat of branes: } \Sigma(Z(\Sigma\mathbb{B}))}$

Corollary: If $\mathcal{B}$ is slightly degenerate, then $Z\Sigma\mathcal{B}$ is a nondegenerate braided fusion 2-category with

$$\Omega Z\Sigma\mathcal{B} \cong \mathrm{SVec}.$$

$$\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(\mathbb{1})$$

Notation: Simple objects of SVec are $1 := \mathbb{C}^{1|0}$ and $e := \mathbb{C}^{0|1}$. the "electron"

Theorem: There are exactly two* of these: "S" and "T".

* up to non-unique iso.

They are almost the same.

The only difference is in the higher braiding data for m, m'. Dif. classes in $H^5(K(\mathbb{Z}_2,2);\mathbb{C}^\times) \cong \mathbb{Z}_2$.

Simple objects:

$1, e \curvearrowright \mathbb{1}$ $\qquad$ $1, e \curvearrowright m$

$\hom(1,c) = \hom(c,1) = \mathrm{Vec}$ $\qquad$ vec

$1, e \curvearrowright c$ $\qquad$ $1, e \curvearrowright m'$

$\simeq \Sigma\mathrm{SVec}$. $\qquad$ the magnetic strings. $\exists$ automorphism switching $m \leftrightarrow m'$.

Fusion rules:

$$c^2 \simeq m^2 \simeq m'^2 \simeq 1$$

Braidings:

Eg: $\mathcal{J} \simeq Z(\Sigma \mathrm{SVec})$. The monoidal map $Z(\Sigma \mathrm{SVec}) \to \Sigma \mathrm{SVec}$

$\mathbb{1}$	$\mapsto$	$\mathrm{Cliff}(0)$
c	$\mapsto$	$\mathrm{Cliff}(1)$
m	$\mapsto$	$\mathrm{Cliff}(0)$
m'	$\mapsto$	$\mathrm{Cliff}(1)$
1	$\mapsto$	$\mathbb{C}^{1\|0}$
e	$\mapsto$	$\mathbb{C}^{0\|1}$

Suppose $Z(\Sigma \mathcal{B}) \simeq \mathcal{J}$. Consider the multifusion 2-cat $\mathcal{A} := \Sigma \mathcal{B} \otimes_{\mathcal{J}} \Sigma \mathrm{SVec}$.

① $Z(\mathcal{A}) = \mathrm{triv}$.

② $\mathcal{A}$ is fusion:

(2.a) A vertex op. in $\mathcal{A}$ is a line in $\mathcal{J}$ ending on the two boundaries.

(2.b) But e does not end on the $\Sigma \mathrm{SVec}$ boundry.

$$\boxed{\text{①+②} \Rightarrow \mathcal{A} \simeq \Sigma \mathcal{C} \text{ for } \mathcal{C} \text{ a nondeg braided fusion 1-cat!}}$$

③ $\mathcal{C}$ is an MME of $\mathcal{B}$.

(3.a) Lines in $\mathcal{C}$ = ways a surface in $\mathcal{J}$ can end on the boundaries.

(3.b) Endings of $\{\mathbb{1}, c\} = \mathcal{B} \subseteq \mathcal{C}$.
Endings of $\{m, m'\}$ = Lines in $\mathcal{C}$ that braid nontriv. w/ $e \in \mathcal{B}$.

Repeat from last slide [KLWZZ]:
If $Z(\Sigma\mathcal{B}) \simeq Z(\Sigma\mathrm{SVec}) =: \mathcal{S}$, then $\mathcal{B}$ has a MME.

So it suffices to show that $\mathcal{T} \not\simeq Z(\Sigma\mathcal{B})$.

aka "absolute"

Idea: $Z(\Sigma\mathcal{B}) \simeq$ operators in a nonanomalous 4D TQFT.

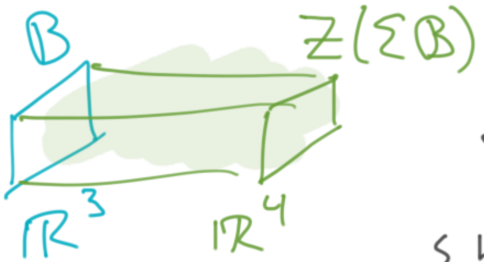
$\mathcal{T} \simeq$ operators in a gravitationally anomalous 4D TQFT.

The anomaly is the 5D invertible TQFT $(-1)^{w_2 w_3}$.

Problem: This anomaly is for oriented TQFTs.

A priori, $Z(\Sigma\mathcal{B}) \leftrightarrow$ non anom. framed TQFT.

Background thm: (braided) fusion n-cat $\leftrightarrow$ $n+2(+1)$-dualizable.
nondegenerate $\leftrightarrow$ invertible.



$\circlearrowleft SO(3)$ action on $\left\{ \begin{array}{c} \text{bulk-boundary} \\ \text{systems} \end{array} \right\}$.

"Oriented TQFT"

this is data: in higher cats, must say how it is fixed

= fixed point for $SO(3)$-action.

= ribbon str on $\mathcal{B}$.

<u>Defn</u>: $\mathcal{B}$ is <u>pseudounitary</u> if it has a (unique!) ribbon structure $b \mapsto \theta_b$ s.t. $q\dim = FP\dim$.

$:= \Big[\theta_b =$  $\Big]$

E.g.: e has spin $\theta_e = -1$.

"spin-statistics relation"

Induces $SO(3)$-fixed structure on braided fusion 2-cat $Z(\Sigma\mathcal{B})$.

<u>Nonanomalous</u>: Partition fn of 5D $SO(3)$-cobd TQFT is $+1$.

Theorem: Up to equivalence, $\mathcal{T}$ has **three** $SO(3)$-fixed strs.

(a) One is anomalous. ← the "canonical"/physical one.

(b) One violates spin-statistics: Induced ribbon str. on $\Omega\mathcal{T} = \mathrm{SVec}$ has $\theta_e = +1$.

(c) One "bad" one.

If $\mathcal{T} = Z(\Sigma\mathcal{B})$, then $\mathcal{T} \to \Sigma\mathcal{B}$ must be $SO(3)$-equiv for one of these.

If I knew what a unitary braided fusion 2-cat was, I think this would be the only unitarizable $SO(3)$-fixed str.

Pf: (1) Construct (a). Show it is fixed by all of $\mathrm{Aut}(\mathcal{T})$.

(2) Then $SO(3)$-strs $\longleftrightarrow$ $\hom(BSO(3), B\mathrm{Aut}(\mathcal{T}))$.

N.B: Uses detailed comp. of $\mathrm{Aut}(\mathcal{T})$.

Explicit description of $\mathcal{T} \Rightarrow$ 5D TQFT of $\mathcal{T}$ is

$$M^5 \mapsto \sum_{\substack{\alpha,\beta \\ \alpha\in H^2(M;\mathbb{Z}_2) \\ \beta\in H^3(M;\mathbb{Z}_2)}} (-1)^{\int_M \alpha\beta + Sq^2 Sq^1\alpha + Sq^2\beta} = (-1)^{w_2 w_3}.$$

How to rule out the "bad" one?

Given an $SO(3)$ structure on a braided fusion 2-cat $\mathcal{A}$, it makes sense to talk about relative $SO(2)$ structures on objects $x \in \mathcal{A}$.

Ambient $SO(3)$ str: Can $\int$ objects on framed (+ normally-framed) surfaces in $\mathbb{R}^3$ ($\mathbb{R}^4$).

Relative $SO(2)$ str on x: Can $\int x$ on oriented surfaces.

If x is simple, rel. $SO(2)$ strs, if they exist, are a torsor for $\mathbb{C}^\times$.

Defn: A relative $SO(2)$ str on x is stable if $\int_{S^2} x = +1$.

$\{$Stable rel. $SO(2)$ strs$\}$ is a $\mathbb{Z}_2$-subtorsor of $\{$rel $SO(2)$-strs$\}$.

Example: In $\mathcal{A} = \Sigma\mathbb{B}$, objects = sep. assoc. algs $\in \mathbb{B}$.

- rel $SO(2)$ str = sym. Frob. str.
- stable: $tr(1)^2 = qdim(A)$.

Theorem [Schaumann, in different language]:

If $\mathbb{B}$ is pseudounitary, then every simple $x \in \Sigma\mathbb{B}$ has a (canonical) positive stable $SO(2)$-str and a negative one.

Corollary: Choose any stable relative $SO(2)$-str on x. The induced stable rel. $SO(2)$ str on x^2 is the positive one.

they do exist

BUT: In the "bad" $SO(3)$ str on $\mathcal{T}$, pick either stable rel. $SO(2)$-str on m. Calculate: the induced stable rel. $SO(2)$-str on $m^2 \simeq \mathbb{1}$ is the negative one.

Conclusion: There is no $SO(3)$-equiv. map $\mathcal{T} \to \Sigma\mathbb{B}$.

So $\mathcal{T} \not\simeq Z(\Sigma\mathbb{B})$. $\blacksquare$