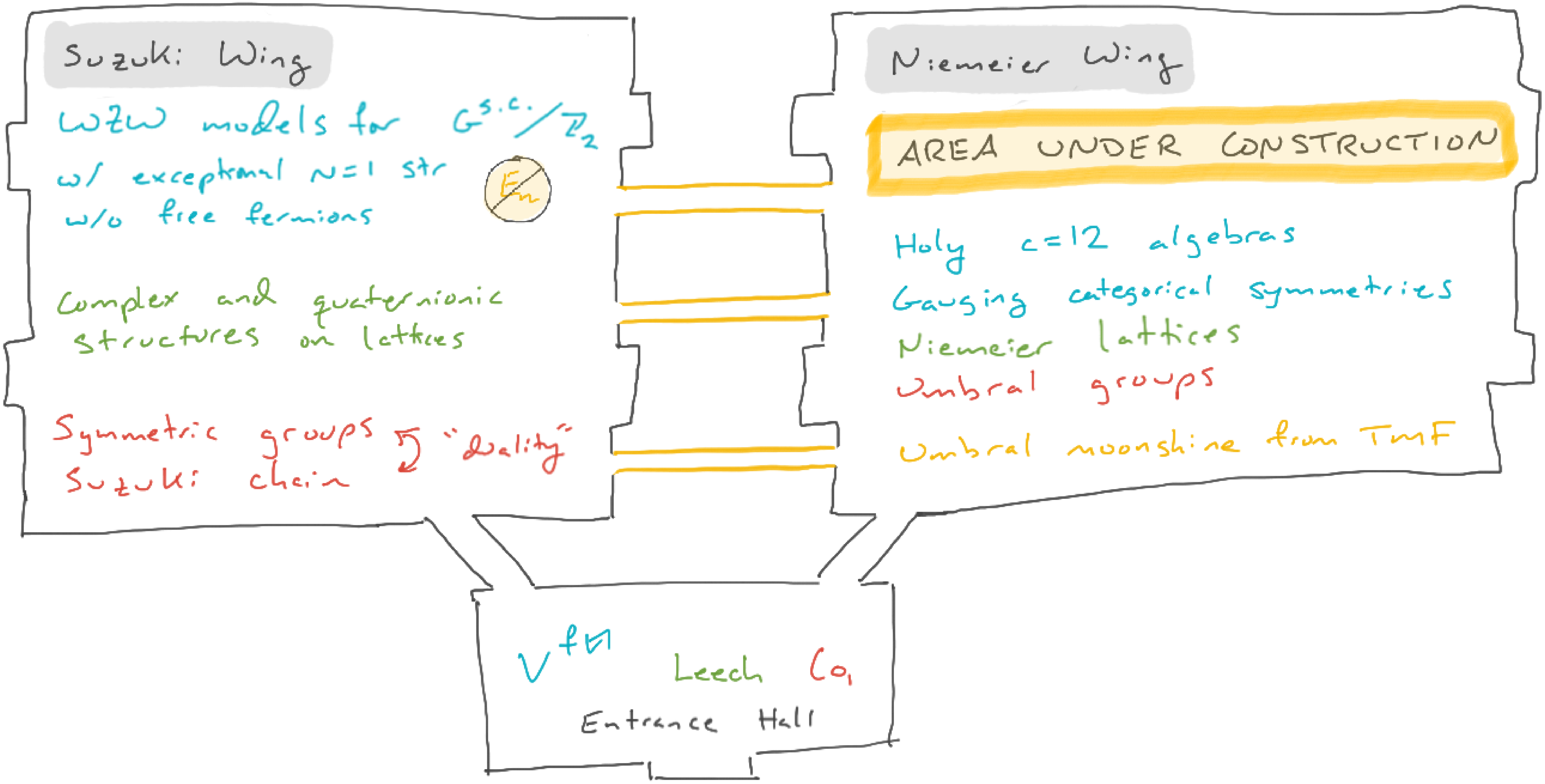
A menagerie of $N=1$ SVOAs

NAAP group, Kavli IPMU, 6 Oct 2021

Theo Johnson-Freyd, Dalhousie & Perimeter

these slides: categorified.net/NAAP.pdf

MUSEUM DIRECTORY



Suzuki Wing

WZW models for $G^{s.c.}/\mathbb{Z}_2$
w/ exceptional $N=1$ str $\cancel{E_n}$
w/o free fermions

Complex and quaternionic structures on lattices

Symmetric groups ↺ "duality"
Suzuki chain

Niemeier Wing

AREA UNDER CONSTRUCTION

Holy $c=12$ algebras
Gauging categorical symmetries
Niemeier lattices
Umbral groups

Umbral moonshine from TMF

$V^{f\natural}$ Leech Co_1

Entrance Hall

Tour of the museum's Suzuki Wing

Theorem. *Suppose V is an $N{=}1$ SVOA with no free fermions, and that $V_{\mathrm{ev}} = G_k$ is a simply connected WZW algebra which is not one of $\mathrm{E}_{7,2}$, $\mathrm{E}_{7,1}^2$, or $\mathrm{E}_{8,2}$. Then V is on the following list:*

V_{ev}	$\dim V_{3/2}$	c	$\mathrm{Aut}_{N=1}(V)$
$\mathrm{Spin}(m)_3$	$\frac{m(m-1)(m+4)}{6}$	$\frac{3m(m-1)}{2(m+1)}$	S_{m+1}
$\mathrm{Spin}(m)_1^3$	m^3	$\frac{3m}{2}$	$\begin{cases} 2^{2(m-1)}{:}(S_3 \times S_m), & m \neq 4 \\ 2^6{:}3S_6, & m = 4 \end{cases}$
$\mathrm{Sp}(2{\times}3)_2$	84	7	$\mathrm{U}_3(3){:}2$
$\mathrm{Sp}(2{\times}3)_1^2$	196	$8\frac{2}{5}$	$\mathrm{J}_2{:}2$
$\mathrm{SU}(6)_2$	175	$8\frac{3}{4}$	$\mathrm{M}_{21}{:}2^2$
$\mathrm{Sp}(2{\times}6)_1$	429	$9\frac{3}{4}$	$\mathrm{G}_2(4){:}2$
$\mathrm{SU}(6)_1^2$	400	10	$\mathrm{U}_4(3){:}D_8$
$\mathrm{Spin}(12)_2$	462	11	$\mathrm{M}_{12}{:}2$
$\mathrm{SU}(12)_1$	924	11	$\mathrm{Suz}{:}2$
$\mathrm{Spin}(12)_1^2$	1024	12	$2^{10}{:}\mathrm{M}_{12}{:}2$
$\mathrm{Spin}(16)_1 \times \mathrm{Spin}(8)_1$	1024	12	$2^8 \cdot \mathrm{O}_8^+(2).2$
$\mathrm{Spin}(24)_1$	2048	12	Co_1

Definition: A simply connected WZW algebra is a unitary VOA V generated by $V_1 :=$ its operators of spin 1.

$\overset{\hookrightarrow}{}$ = conformal dimension

$$V \overset{!!}{=} G_k$$

For any VOA, V_1 is a Lie algebra with a (negative-definite) nondeg. sym pairing k satisfying an integrality condition.

Set $G =$ (compact form of) simply connected gp w/ Lie alg V_1.

Then (G, k) determines the WZW alg uniquely.

$\hookrightarrow$ k is called the "level". It lives in $H^4(BG; \mathbb{Z})_{>0}$.

An SVOA is a VOA object $V = V_{ev} \oplus V_{odd}$ in super vector spaces.

Then V_{ev} is a VOA, and V_{odd} is a V_{ev}-module s.t.

$V_{odd} \boxtimes_{V_{ev}} V_{odd} \simeq V_{ev}$ (assuming $V_{odd} \neq 0$).

N.B: $(-1)^F = (-1)^{2 \cdot \text{spin}}$.

Definition: A module M for a VOA V is an abelian anyon if it is invertible for $\boxtimes_V$.

Theorem (Fuchs): If $V = G_k$ is a simply connected WZW alg, its abelian anyons form a copy of the group $Z(G)$ except for $E_{8,2}$, where $Z(E_8) = \{1\}$ but abelian anyons $= \mathbb{Z}_2$.

Definition (Henriques): A non-simply connected WZW alg G_k, if it exists, is a VOA of the form $G_k^{s.c.} \oplus \left(\begin{smallmatrix}\text{abelian anyons}\\ \text{corresponding to } Z(G)\end{smallmatrix}\right)$.

I.e. it is a simple current extension of a simply connected WZW alg (uncontaminated by $E_{8,2}$).

In particular, our SVOA $G_k \oplus V_{odd}$ is a WZW alg for $G/\mathbb{Z}_2$

Definition: An $N=1$ structure on an SVOA is a choice of operator "τ" of spin $= 3/2$ s.t.

↳ hence odd under $(-1)^F$

$$\tau(z)\,\tau(0) \sim \frac{\frac{2}{3}c}{z^3} + \frac{\nu(0)}{z}$$

the conf. vector, often called T.

The spin-1/2 operators are (superpartners of) the Lie subalg of the spin-1 ops that commute with τ. ← this constrains V_{odd} as a V_{ev}-module!

"No spin-1/2 ops" $\Longleftrightarrow$ $\mathrm{Aut}_{N=1}(V)$ is finite.

Since τ is odd, it is $\mathbb{Z}$-moded in the Ramond sector and $\tau_0^2 = L_0 =$ "spin". So all Ramond sector V-modules must have characters that are indep. of q.

$\mathrm{Tr}\left((-1)^F q^{L_0}\right)$"

↑ this highly constrains V.

The map of those G_k s.t. $\exists$ $\mathbb{Z}_2$-anyon of spin $= 3/2$ and all R-sector characters are indep of q:

↑ Li gives a general formula for spins of ab. anyons of WZW algs.

these don't seem to fit. → Ignore them!

magic ↓

$$\begin{tikzcd}
 & & E_{8,2} & & & & & \\
 & & E_{7,2} \arrow[u, hook] \arrow[r, hook] & E_{7,1}^2 & & & & \\
 & & \mathrm{Spin}(12)_2 \arrow[u, hook] \arrow[r, hook] \arrow[dr, hook] & \mathrm{Spin}(12)_1^2 \arrow[u, hook] \arrow[rr, hook] & & \mathrm{Spin}(24)_1 & & \\
 & \mathrm{SU}(6)_2 \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \mathrm{SU}(6)_1^2 \arrow[r, hook] \arrow[ur, hook] & \mathrm{SU}(12)_1 \arrow[urr, hook] & & & & \\
\mathrm{Sp}(2\times3)_2 \arrow[ur, hook] \arrow[r, hook] & \mathrm{Sp}(2\times3)_1^2 \arrow[r, hook] \arrow[ur, hook] & \mathrm{Sp}(2\times6)_1 \arrow[ur, hook] & & & \mathrm{Spin}(16)_1 \times \mathrm{Spin}(8)_1 \arrow[uu, hook] & & \\
\mathrm{Sp}(2\times1)_2^3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Sp}(2\times1)_1^6 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Sp}(2\times2)_1^3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{SU}(4)_1^3 \arrow[uu, hook] \arrow[rr, hook] \arrow[d, equal] & & \mathrm{Spin}(8)_1^3 \arrow[u, hook] \arrow[d, equal, "\text{triality}"] & & \\
\text{“}\mathrm{Spin}(3)_1^3\text{”} \arrow[r, hook] & \mathrm{Spin}(4)_1^3 \arrow[r, hook] & \mathrm{Spin}(5)_1^3 \arrow[r, hook] & \mathrm{Spin}(6)_1^3 \arrow[r, hook] & \mathrm{Spin}(7)_1^3 \arrow[r, hook] & \mathrm{Spin}(8)_1^3 \arrow[r, hook] & \mathrm{Spin}(9)_1^3 \arrow[r, hook] & \cdots \\
\text{“}\mathrm{Spin}(3)_3\text{”} \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(4)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(5)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(6)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(7)_3 \arrow[u, hook] \arrow[r, hook] & \mathrm{Spin}(8)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal, "\text{triality}"] & \mathrm{Spin}(9)_3 \arrow[u, hook] \arrow[r, hook] & \cdots \\
\mathrm{Sp}(2\times1)_6 \arrow[r, hook] & \mathrm{Sp}(2\times1)_3^2 \arrow[r, hook] & \mathrm{Sp}(2\times2)_3 \arrow[r, hook] & \mathrm{SU}(4)_3 \arrow[rr, hook] & & \mathrm{Spin}(8)_3 & &
\end{tikzcd}$$

↙ systematic

But how to construct τ? How to prove uniqueness?

(1) Infinite families $\mathrm{Spin}(m)_1^3/\mathbb{Z}_2$ and $\underbrace{\mathrm{Spin}(m)_3/\mathbb{Z}_2 =: SO(m)_3}_{\text{e.g. unique } A_{m+1}\text{-invariant } \tau.}$:

just write down the answer.

(2) The others are:

$$\left.\begin{array}{lll} \bullet\ \mathrm{Spin}(m)_k & \text{s.t.} & m\cdot k=24 \\ \bullet\ SU(m)_k & \text{s.t.} & m\cdot k=12 \\ \bullet\ Sp(2\times m)_k & \text{s.t.} & m\cdot k=6. \end{array}\right\}$$

these all embed into $\mathrm{Spin}(24)_1$, and in all cases the coset is exceptionally iso to $\mathrm{Spin}(m)_3$!

(3) Theorem (Duncan): $\mathrm{Spin}(24)_1/\mathbb{Z}_2 =: V^{f\natural}$ has unique $N=1$ str.

$\mathrm{Aut}_{N=1}(V^{f\natural}) = Co_1 := \mathrm{Aut}(\text{Leech lattice})/\{\pm 1\} \subseteq PSO(24).$

Study conf. embedding

$$\begin{tikzcd} (G_k/\mathbb{Z}_2)\times SO(m)_3 \arrow[r, hook] \arrow[d, squiggly] & V^{f\natural}. \arrow[d, squiggly] \\ A_{m+1} & Co_1 \end{tikzcd}$$

Look for τ fixed by centralizer of $A_{m+1} \subset Co_1$.

The Suzuki Chain

Thompson, Suzuki:

$$\begin{tikzcd}
A_9 \arrow[rr, hook] & & Co_1 \\
\cap & & \cap \\
PSO(8) \arrow[r, "\simeq", "\text{triality}"'] & PSO(8) \arrow[r, hook] & PSO(24)
\end{tikzcd}$$

The Suzuki Chain of subgps of Co_1 are the centralizers of subgps of this A_9.

E.g.:

- $A_3 = \mathbb{Z}_3 \subset A_9$ selects a complex structure on the Leech lattice, centralized by Suzuki sporadic gp.

  ↳ i.e. makes it into a lattice over $\mathbb{Z}[\frac{-1}{2} + \frac{\sqrt{-3}}{2}]$.

- A_4 selects a quaternionic str centralized by $G_2(\mathbb{F}_4)$.
- etc.

Tour of the museum's Niemeier wing

Conway, Sloan: The Leech lattice has 23 holy sublattice:

choose a deep hole λ_0, and look at

i.e. a vector in $\mathbb{R}^{24} = \mathbb{R} \otimes \text{Leech}$ of maximal distance from all Leech vectors

$$L := \{ \lambda \in \text{Leech s.t. } \langle \lambda, \lambda_0 \rangle \in \mathbb{Z} \}$$

$L \subset \text{Leech}$ is full rank, and $\text{Leech}/L \simeq \mathbb{Z}_m$

← coxeter # of the hole.

Conjecture: $V^{f\natural}$ has 23 "holy $N=1$ subalgebras":

- $L \subset \text{Leech}$ is full rank $\Longleftrightarrow$ $V \subset V^{f\natural}$ is conformal.
- $V^{f\natural}/V = \mathrm{Rep}(SU(2)_{m-2})$
- naturality / functoriality for Aut(hole) = "umbral group".

By "$V^{f\natural}/V$" I don't mean the SVOA-theoretic coset, which is trivial since $V \subset V^{f\natural}$ is conformal. Rather,

$\dfrac{V^{f\natural}}{V} :=$ the fusion category of line operators in $V^{f\natural}$ which commute w/ V.

Conversely,

$V =$ fixed points for an action of this fusion cat on $V^{f\natural}$ by "global categorical symmetries".

So I am asking for "holy actions" of $\mathrm{Rep}(SU(2)_{m-2})$ on $V^{f\natural}$.

Why $\mathrm{Rep}(SU(2)_{m-2})$? Suppose m is prime. Then this cat has a reduction mod m which is symmetric, and agrees w/ "$\mathrm{Rep}_{\mathbb{F}_m}(\mathbb{Z}_m)$-semisimplified".

I can build a few superconformal subalgebras $V \subset V^{f\natural}$ with the correct representations and automorphisms, but the constructions are ad hoc modifications of "Suzuki-wing" objects, and I don't know if the results are holy.

Example: For the deep hole named "$6A_4$",

Start w/ $SO(4)_3 \times (Sp(2\times 3)_1^3 / \mathbb{Z}_2) \subset V^{f\natural}$

$Aut = S_5 \times (J_2 \rtimes \mathbb{Z}_2)$

Condense some anyons to get $(SU(2)_3 \times Sp(2\times 3)_1 \times Spin(12)_1) \oplus \text{odd} \subset V^{f\natural}$.

breaks auto gp to $A_5 \times J_2$.

Pick an element in J_2 of conj. class 2B. Take fixed pts.

breaks auto gp to $A_5 \times (2^4 \rtimes \underbrace{A_5})$.

This subgp is the "umbral gp".
Discard the rest of Aut.

WHY? There is a way to "gauge" categorical symmetries (with the correct "anomaly cancellation data).

antiholo version

$$\overline{V}^{f\natural} \times \left(\mathcal{N}=(0,1) \text{ WZW model for } SU(2) \text{ with levels } (m-2, m) \right)$$

$\circlearrowleft$ holy $\qquad\qquad$ $\circlearrowleft$ verlinde lines

$$\underbrace{\mathrm{Rep}(SU(2)_{m-2}) \qquad \mathrm{Rep}(SU(2)_{m-2})}_{\text{"anomalies cancel."}}$$

$\rightsquigarrow$ full $\mathcal{N}=(0,1)$ SCFT w/ $(c_L, c_R) = (1 - \frac{6}{m}, 14\frac{1}{2} - \frac{6}{m})$.

Expectation: $\mathcal{N}=(0,1)$ SCFTs provide cocycles for TMF.

for the umbral gp

Conjecture: These "holy SCFTs" are equivariantly nullhomotopic in TMF (in fact, in Tcf).

If so, the nullhomotopy would "explain" umbral moonshine.