A deformation invariant of 2D SQFTs

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Based on 1904.05788, jt w/ D. Gaiotto, [redacted]
and 1902.10249, jt w/ D. Gaiotto and E. Witten

As I mentioned already yesterday, one of my main goals is to understand the algebraic topology of the space

$$\{\underbrace{N=(0,1)}_{\text{"minimal susy"}}\ 2D\ SQFTs\}$$

Physicists have a good understanding of the meaning of these words, but let me emphasize that I don't claim a mathematical defn. In addition to figuring out the general defn of "QFT", we also need some analytic words, because I want QFTs which are

- unitary (so that I can Wick-rotate)
- "compact" at least in a weak sense
  - roughly: $\hat{H}$ should have nice spectrum. In the QM case, the defn is that $\exp(-t\hat{H})$ should be traceclass for $t>0$. This is slightly stronger than "compact resolvent".

And of course then we need to topologize this space.

Before talking about the SUSY specifically case, let me say a few words about ~~non-susy~~ 2D QFTs w/ arbitrary SUSY.

- {2D QFTs} is infinite dim'l. Probably not a manifold, but ∞-dim "tangent bundle" nonetheless. (Top'l feels like × ∞-dim.)
- It carries a "morse function", c. Downward flow of c is called "RG flow". Every pt has finite morse index.
- c is at best morse-bott. $crit(c) = \{CFTs\}$ and this is expected to be finite dim.
- $\{RCFTs\} \subseteq \{CFTs\}$, topologically $\approx \mathbb{Q} \subseteq \mathbb{R}$.
  ↑ do have a math-natural defn.
- $\{\text{holomorphic CFTs}\} \subseteq \{RCFTs\}$. discrete pts.

Assuming downward flows converge ("IR-complete"), then we should be able to explore the whole space by "zig-zagging along RG flow lines".

- topologized eg. by scanning up the spectrum: thys w/ "similar" IR are "near".

Actually, a 2D QFT has two central charges c_L, c_R, w/ $c = \frac{c_L + c_R}{2}$ the morse fn. The grav. anomaly

$$c_R - c_{~~R~~L} \in \tfrac{1}{2}\mathbb{Z}$$

is an RG-flow invariant.

$$\{2D\ QFTs\} \longrightarrow \tfrac{1}{2}\mathbb{Z}.$$

This enhances to a total anomaly map $K(\mathbb{Z},4)\cdot K(\mathbb{Z}_2,2)\cdot K(\mathbb{Z}_2,1)\cdot K$

<u>Expectation</u>: In the absence of SUSY,
$\{2D\ QFTs\} \xrightarrow{\text{anomaly}} (-)$ is a homotopy equiv.
i.e. the <u>only</u> def. inv. is the anomaly.

OK, now the SUSY case. BTW, "minimal SUSY" aka "$N=(0,1)$" means the following. In addition to the Poincare symmetries $\hat{H}=\partial_t$, $\hat{P}=\partial_x$, there is a <u>right-moving susy</u> $\hat{G}$ s.t. $\hat{G}^2=\hat{H}-\hat{P}=\partial_{\bar{z}}$.
<u>Wick-rotate</u>: $z, \bar{z}$.

Given an $N=(0,1)$ SQFT $\mathcal{F}$, here are some natural questions:

(1) Is SUSY <u>spontaneously broken</u> in $\mathcal{F}$?
The far IR of an SQFT is just the susy ground states. So this is asking:
Does $\mathcal{F} \longrightarrow \varnothing$ under RG flow?

(2) Does $\mathcal{F}$ admit an inf. deformation (by a susy-preserving operator) s.t. the deformed thy has spont. susy breaking?

(3) Can $\mathcal{F}$ be connected to an sqft ~~...~~ ~~...~~ w/ spont susy breaking ~~to~~ via a path in $\{2D\ SQFTs\}$?

Here's a nontrivial example of (1, 3).
<u>Example</u> [Gaiotto, JF, Witten]:
Consider the $N=(0,1)$ sigma model w/ target the round S^3. ~~...~~

<u>This means</u>: There is a scalar boson $\hat{x}$ valued in S^3, and a right-moving fermion spinor valued in T_{S^3}.
In the quantum theory, there is an anomaly to patching the spinor over S^3. ~~[illegible]~~ To cancel the anomaly requires choosing a "B-field", which in math is called a "geometric string str." In general, there might be an obstruction, and the set of B-fields, if nonempty, is merely a <u>torsor</u> for $H^3(M, \mathbb{Z})$. In the S^3 case this torsor has a base pt: the unique B-field fixed under reflection. The ~~set~~ choice $K \in H^3(S^3; \mathbb{Z}) = \mathbb{Z}$ is called "K units of H-flux".

In any case, consider the σ-model S^3_K. What is its IR limit?

<u>Answer</u>: $S^3_{K\,(0,1)} \longrightarrow$ (0,1) WZW for $SU(2)$ w/ KM levels $|K|-1$, $|K|+1$. (if $K \neq 0$)

<u>"Pf"</u>: • Perturbation thy in $|K| \gg 0$.
• anomaly matching.

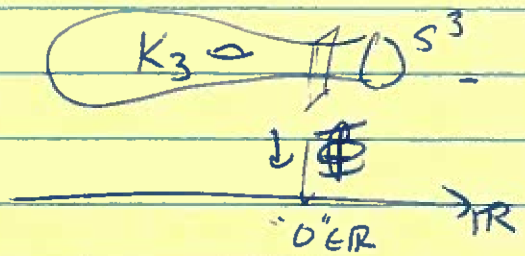
N.B.: $K>0$ vs $K<0$ differ by flip of orf.

$\boxed{\text{When } K=0}$: $S^3_0 \to \emptyset$. <u>Spont. susy breaking!</u>

So there's an example of (1).

Let me now give an example of (3):

Consider the $K3$ surface, minus one pt.



The $K3$ has a unique B-field, and it restricts to $k=24$ near the S^3.

Now consider turning on a Lagrange multiplier to force $f=0$. The way you do this is:

- add a left-moving fermion λ.
- add a superpotential $W = \lambda \Phi(x)$
  meaning $\tau \rightsquigarrow \tau + \omega$.

The ~~[illegible]~~ low-energy behavior is the just $\Phi = 0$. i.e. the S^3.

But now change $f \rightsquigarrow f + r$ for $r \to -\infty$.
You get a family of things. When $r \ll 0$, get the empty sigma model.

Conclusion: $S^3_{k=24}$ has property (J).

More generally: ~~[illegible]~~

$$\underbrace{MString}_{\substack{\text{cob. spectrum} \\ \text{for manifolds w/ } B\text{-field}}} \longrightarrow \{SQFTs\}.$$

Suppose, OTOH, you want to protect an SQFT from having (3). You would need a def. inv.

There is a famous one:

<u>Defn:</u> Given $\mathcal{F}$, the <u>elliptic index</u> is

$$Z_{\mathcal{F}}(\tau,\bar{\tau}) = \text{partition fn on tori w/ non-bounding spin str.} \quad \times\, q^{(c_R - c_L)/24}$$

to correct an ambiguity

(actually, this also depends on the worldsheet volume.)

It is manifestly real-analytic modular.

(I) <u>Standard fact:</u> $Z_{\mathcal{F}}(\tau,\bar{\tau})$ is holomorphic.

<u>Pf:</u> $\frac{\partial}{\partial \bar{\tau}} Z$ = one-pt fn of $T_{\bar{z}\bar{z}}$

= one-pt fn of $\hat{G}[\hat{G}]$.

In the non-bounding spin str, a one-pt fn of $\hat{G}[\hat{X}] = 0$ for any $\hat{X}$.

(II) <u>Standard fact</u> q-expansion of $Z(\tau)$ is integral.

<u>Pf:</u> Break manifest modularity and compactify on an S^1. Get an S^1-equiv. SQM model. q-expansion counts ~~…~~ ground states.

reason: $\langle [G, \mathcal{X}] \rangle = 0$ requires Stokes thm.

Actually, (I) only holds for compact thys!

Defn: An SQFT $\mathcal{X}$ has cylindrical ends $\mathcal{Y}$ if $\exists$ a (self-adjoint) operator $\mathcal{X} \to \mathbb{R}$ s.t. $\mathcal{X}_r := \mathcal{X} + \lambda + (W = \lambda r)$ is a fmly of compact thys, w/ spont. sym breaking at $r \ll 0$ and $\mathcal{X}_r \to \mathcal{Y}$ stabilizing for $r \gg 0$.

Claim: [Gaiotto-JF]. Suppose $\mathcal{X}$ has cylindrical end $\mathcal{Y}$ and grav anomaly $2(c_R - c_L) \in 4\mathbb{Z}$. Then the elliptic genus of $\mathcal{X}$ has an holomorphic anomaly

$\tau_2 := \frac{\tau - \bar{\tau}}{2i}$

$$\sqrt{-8\tau_2}\,\frac{\partial Z_{\mathcal{X}}}{\partial \bar{\tau}} = \overset{\text{normalization factor: } \sqrt{i}^{\#}\eta^{\#}}{(\quad)} \cdot \langle \text{one pt fn of } \hat{G} \text{ in } \mathcal{Y} \rangle.$$

Justification: (1) Stoke's thm in field space.
(2) we checked carefully many examples. to pin the normalization.

(II) essentially does hold (up to an explicit correction related to APS η-fn / "mod-2 index").
To understand it, go back to the pf: we broke manifest modularity to place on an S^1; this was really

$$\lim_{\bar{\tau} \to -i\infty} \hat{f}(\tau, \bar{\tau}) =: f(\tau)$$

$\hookleftarrow$ this is the thing

Summary: Suppose you have a thy $\mathcal{Y}=\mathcal{Y}_{+\infty}$ and ~~a family of~~ family $\mathcal{Y}_r$, $r\in\mathbb{R}$ defining I to $\emptyset$, i.e. $\mathcal{Y}_{-\infty}=\emptyset$.

how {SQFTs} is a spectrum.

* This is the same as "$\mathcal{Y}$ couples to a nondynamical boson "r"".
* Dynamicalize $r \rightsquigarrow \varphi$. Get the superpartner ψ, changing the grav. anomaly by 4Z.

"$\mathcal{Z}$" = "$\int \mathcal{Y}_r\, \mathcal{D}r$."

Then $\hat{f}(\tau,\bar\tau) = Z(\mathcal{X})$ solves:

(0) $\hat{f}$ is real analytic modular.

(1) $\sqrt{-8\tau_2}\,\dfrac{\partial\hat{f}}{\partial\bar\tau} = g(\tau,\bar\tau) =$ one-pt fn of susy of $\mathcal{Y}$.

(2) $f(\tau) = \lim\limits_{\bar\tau\to -i\infty} \hat{f}$ has integral q-series.

Conversely, given $g(\tau,\bar\tau)=$ one-pt fn for $\mathcal{Y}$, there is an obstruction to finding such $\hat{f}$.

Indeed: ~~[illegible]~~

(0,1) g determines $[\hat{f}] \in \dfrac{\text{real analytic modular forms}}{MF = \text{holo. modular forms}}$

(2) hence g determines $[f] \in \dfrac{\mathbb{C}[[q]]}{MF}$.

But this class must be $\in \mathbb{Z}[[q]]$. So

Conclusion: $[f] \in \dfrac{\mathbb{C}(q)}{MF + \mathbb{Z}(q)}$ is the obstruction.

i.e. this class is (1) defined by $\mathcal{Y}$.
(2) a def. inv.

<u>Example:</u> Recall $S^3_k \rightsquigarrow WZW\ (k-1, k+1)$.

This is an RCFT, so we have good ability to compute. E.g.: when $k=1$, this is $Fer(3)$, $G = :\psi_1\psi_2\psi_3:$

~~One-pt fn is~~

$$\cancel{g(\tau,\bar\tau) = \eta^3(\bar\tau) \quad \text{if} \quad k=1}$$

In general, $G = \sqrt{-1}\sqrt{\frac{2}{k+1}}\, :\psi_1\psi_2\psi_3:$

$$+ \# \sum \psi_a J^a \ ] \leftarrow \text{with correct normaliz.}$$

One-pt fn is

$$\langle G\rangle = g(\tau,\bar\tau) = \sqrt{\frac{2}{k+1}}\ \eta(\bar\tau)^3 \cdot \underbrace{Z^{\text{bosonic WZW}}_{k-1}(\tau,\bar\tau)}_{\sum\ ||}$$

$$= 4(k+1)\sqrt{2(k+1)}\sum_{2j+1=1}^{k}\left|\Theta_{k+1,2j+1}(\tau)\right|^2.$$

An explicit soln to ~~[illegible]~~ eqn (1) is given in Harvey-Murthy-Nazaroglu.
Its holomorphic part is, after much computation,

$$f(\tau) = k\,\underbrace{G_2(\tau)}_{\substack{\text{Eisenstein series} \\ -\frac{1}{24}+\mathbb{Z}[[q]]}} + \mathbb{Z}[[q]].$$

this is nonzero ~~has order~~ ~~"$-k\cdot\frac{1}{24}$"~~ in $\frac{\mathbb{C}[[q]]}{MF + \mathbb{Z}[[q]]}$!
[illegible] $\frac{1}{24}\mathbb{Z}$.