# SVOAs and some exceptional groups

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Main classification result:

If $V$ is an $N=1$ SVOA s.t.:
(1) $V_{ev}$ is a s.c. WZW model
(2) $V_{1/2} = 0$.
(3) $V \not\supseteq E_7$ or $E_8$
Then $V$ is one of:

[Noether: $V_{1/2} = \mathrm{Lie}\, \mathrm{Aut}_{N=1} V$.]

[suffice: $V \not\supseteq E_{7,2}$ as a subalg]

| $V_{ev}$ | $\dim V_{3/2}$ | $c$ | $\mathrm{Aut}_{N=1}(V)$ |
|---|---|---|---|
| $\mathrm{Spin}(m)_3$ | $\frac{m(m-1)(m+4)}{6}$ | $\frac{3m(m-1)}{2(m+1)}$ | $S_{m+1}$ |
| $\mathrm{Spin}(m)_1^3$ | $m^3$ | $\frac{3m}{2}$ | $2^{2(m-1)}:(S_3 \times S_m)$ |
| $Sp(3)_2$ | 84 | 7 | $U_3(3):2$ |
| $Sp(3)_1^2$ | 196 | $8\frac{2}{5}$ | $J_2:2$ |
| $SU(6)_2$ | 175 | $8\,{}^3/_4$ | $M_{21}:2^2$ |
| $Sp(6)_1$ | 429 | $9\,3/4$ | $G_2(4):2$ |
| $SU(6)_1^2$ | 400 | 10 | $U_4(3):D_8$ |
| $\mathrm{Spin}(12)_2$ | 462 | 11 | $M_{12}:2$ |
| $SU(12)_1$ | 924 | 11 | $Suz:2$ |
| $\mathrm{Spin}(12)_1^2$ | 1024 | 12 | $2^{10}:M_{12}:2$ |
| $\mathrm{Spin}(16)_1 \times \mathrm{Spin}(8)_1$ | 1024 | 12 | $2^8 \cdot O_8^+(2).2$ |
| $\mathrm{Spin}(24)_1$ | 2048 | 12 | $Co_1$ |

$\hookrightarrow$ enhances to $3S_6$ when $m=4$.

I'll say a bit about the details of the statement in a moment. But what I really want to tell you is why I find it. See, this is the type of really fun project where you get to sit and calculate for awhile. Does it matter?

Probably not. But it was fun.

Why do I care? An ongoing question in ~~gp~~ exceptional group theory is: where do exceptional groups come from? The answer is often somewhat disappointing. Some groups, e.g. $J_3$, don't seem to come from anywhere: Janko just found it by identifying a large subgp. More typically, a sporadic gp is the automorphisms of some clever, but complicated, combinatorial object, e.g. $HS = \mathrm{Aut}$(some highly-symmetric graph on 100 vertices). One sporadic gp, though, is particularly <u>fundamental</u>: Conway's largest gp $Co_1$.

Conway found $Co_1$ as $\mathrm{Aut}(\text{Leech lattice})/\pm 1$. $\subseteq PSO(24)$
But he already showed it is more fundamental: it is
$$\mathrm{Aut}(II_{25,1})/\mathrm{Weyl}((II_{25,1}) = \mathrm{Out}(II_{25,1}).$$

I got interested by yet another appearance of $Co_1$, which is so far essentially unrelated to these lattice ones:

<u>Thm (Duncan):</u>
$$Co_1 = \mathrm{Aut}_{N=1}\Big(\text{unique } N=1 \text{ SVOA w/}$$
- $c = 12$
- no free fermions
- holomorphic, i.e. $\mathrm{Rep}(V) = (s)\mathrm{Vec}$ $\Big)$

$= "V^{f\natural}"$

For ~~this~~ $V^{f\natural}$, $V_{ev} = \mathrm{Spin}(24)_1$.

Let me explain some of these words. Given this conference, I won't try to define VOA (and in any case I really want "nice" ones, e.g. unitary CFT-type, ...). An <u>N=1 structure</u> is a choice of <u>superconformal vector</u> $\tau \in V$ which should be a Virasoro primary of spin 3/2 s.t.
$$\tau(z)\tau(0) \sim \frac{\frac{2}{3}c}{z^3} + \frac{\nu(0)}{z}$$
$\nu$ = conformal vector.

and a s.c. WZW alg is a nice VOA gen by its free bosons.

What I want is for you to think of this as choosing a sort of "differential" with "curvature" = $\nu$.

A free fermion in a SVOA is a field of spin 1/2.
You probably know that the free bosons $V_1$ are a Lie alg, and $V_1 \longrightarrow \mathrm{Der}(V) = \text{Lie Aut}(V)$, and this is iso if $V$ is rat'l.
In ~~the~~ $N=1$ case, $V_{1/2}$ is a Lie alg, and

$$V_{1/2} \longrightarrow \text{Lie Aut}_{N=1}(V), \quad \text{iso if } V \text{ rat'l.}$$

Lie alg str: $[\alpha, \beta]_{N=1} = [\alpha, \tau(\beta)]_{N=0}$.

In addition to having such an elegant defn, $Co_1$ is important because many exceptional gps arise as large subgps of $Co_1$. Thompson ex identified an interesting family. Specifically, $\exists$

$$A_9 \subseteq Co_1$$

Now take an alternating subgp of $A_9$, and compute its centralizer:

$$\begin{array}{ccccc}
A_9 & & & & A_3:2 = S_3 \\
\rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
A_8 & & & & A_4:2 = S_4 \\
\rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
A_7 & & & & L_2(7):2 \\
\rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
A_6 & & & & U_3(3):2 \\
\rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
A_5 & & & & J_2:2 \\
\rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
A_4 & & G_2(4) & & G_2(4):2 \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
A_3 & & 3Suz & \rightsquigarrow & Suz:2 \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
A_2 & & Co_1 & &
\end{array}$$

Or use

$$\begin{tikzcd}
A_4 \times A_4 \arrow[d, squiggly] & \text{or} & A_3 \times A_3 \times A_3 \arrow[d, squiggly] \\
U_4(3):D_8 & & 2^{2+12}:(S_3 \times A_8)
\end{tikzcd}$$

what is this? well,
$$\begin{array}{ccc} A_9 & \subseteq & PSO(8) \\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\ Co_1 & \subseteq & PSO(24) \end{array}$$

~~and~~
But be careful: $A_9 \subseteq PSO(8)$ breaks triality, and the embedding into $PSO(24)$ corresponds to

$24 = 3$ copies of the spin rep of $PSO(8)$.

Then
$$\begin{array}{cccc} A_{m+1} & \subseteq & PSO(m+h) & \\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & \\ A_9 & \subseteq & PSO(8) & \\ \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & \leftarrow \text{spin} \\ Co_1 & \subseteq & PSO(24) & \end{array}$$

so what we're really doing is a fun game w/ coincidences of spin gps. e.g.

$$A_{\cancel{8}}7 \rightsquigarrow Spin(6) \cancel{\leftrightarrow} = SU(4) \subseteq \underset{\text{as } \mathbb{C}^4 \underset{\text{forget } \sqrt{-1}}{=} \mathbb{R}^8}{SO(8)}.$$

Restoring WZW levels, these give some exceptional embeddings
$$Spin(m)_3 \subseteq Spin(24),$$
e.g.
$$Spin(6)_3 = SU(4)_3 \subseteq Spin(24)_1.$$

~~Most, [illegible]~~

Something you can always do w/ a VOA embedding is to compute its coset, which is a type of centralizer. Up to simple current extensions, you get in this way standard level-rank dualities, e.g.
$$\frac{Spin(24)_1}{SU(4)_3} \simeq SU(3)_4.$$

What am I saying? $A' = ~~B/A~~ := \{ b \in B$ s.t. $b(z)a(0) \sim 0 \ \forall a \in A \}$.

It is conformal w/ $\nu_{~~B/A~~ A'} = \nu_B - \nu_A$, automatically on ~~[illegible]~~ $A'$

Now, if $A$, $B$ both superconformal, then let's say $A \hookrightarrow B$ is supersymmetric embedding if

$$\tau_B - \tau_A \in A'.$$

in which case it is automatically a supercurrent.

This is ultimately what's going on under the hood in my classification. I told you that $Spin(m)_3$ ⊕ odd part has a unique susy. And I know some favourite embeddings $Spin(m)_3 \hookrightarrow Spin(24)$, . So I need to work out whether I can make this embedding supersymmetric.

[That the classification is complete uses some other, essentially standard, methods: ~~ess~~ character tables, dimensions, etc.]

But VOAs have another type of centralizer. See, ~~B~~ in examples, it often happens that $A'' = A$ for $A \hookrightarrow B$. (Always $A \subseteq A''$.) But not always. E.g. this completely fails when $A \hookrightarrow B$ is conformal. then $A' = \mathbb{C}$, and $A'' = B$.

E.g. when $A = B^G$ for some finite gp $G$ acting on $B$. more generally, Andre told us that fusion categories act on various things, ~~ae~~ and that list includes VOAs. If $\mathcal{C} \circlearrowright B$, can define $A = \mathcal{C}' = ~~\mathcal{C}~~ B^{\mathcal{C}}$, and conversely if $A \hookrightarrow B$ conformal and nice, then $A = \mathcal{C}'$.

~~So maybe $\mathcal{C} \cong A'$?~~

In a few entries on my list, esp when c's agree, I needed such symmetries. E.g.

$$Spin(12)_1^2 = Spin(24)_1^{\mathbb{Z}/2}$$

for $\mathbb{Z}/2$ ~~[illegible]~~ = reflect in 12 dimensions

$$Spin(8)_1 \times Spin(16)_1 = \text{''}$$

for $\mathbb{Z}/2$ = reflect in 8 (~~[illegible]~~ even 16) dimensions.

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I want to end with something I don't know how to solve. In general, we have pretty good methods to decide if some finite gp acts on some VOA: this is "classical" rep thy of finite gps.

But we have <u>essentially</u> <u>not</u> methods to decide if a Lie cont acts on a given VOA.

I've been after another family of subgps of $Co_1$ called the <u>code</u> or <u>umbral</u> gps. They are well understood from the Leech lattice picture: they are Out(L) for L a Niemeier lattice.

Each Niemeier lattice has a <u>coxeter number</u> h.

In the lattice VOA, you get something like

$$\left(\text{Cyclic of order } h\right) \times (\text{umbral gp}) \circlearrowright V_{Leech}.$$

Now, I'd like to suggest that $V^{f\natural}$ is more "<u>fundamental</u>" than $V_{Leech}$. ~~Anyway~~ And

$$(\cancel{\;\;}) (\mathbb{Z}/h) \rightsquigarrow SU(2)_{h-2}.$$

Other moonshine work also supports this.

<u>Conjecture:</u>
For each Niemeier lattice $L$, there is a distinguished action

$$\underbrace{SU(2)_{h-2}}_{\text{just as a fusion cat}} \times \text{umbral gp} \circlearrowright V^{f\natural}.$$

equiv. a umbral-equiv subcat of $V^{f\natural}$.

E.g. $SU(2)_{2-2} = \text{triv.}$

$SU(2)_{3-2} = \mathbb{Z}/2$ w/ $\overset{\text{nontriv}}{\text{anomaly}}$

$SU(2)_{4-2} = \text{Ising}$, $\hookrightarrow \Big(\underbrace{(V^{f\natural})^{\mathbb{Z}/2}}_{\overset{\shortparallel}{Spin(8)_1 \times Spin(16)_1},}\Big)^{\mathbb{Z}/2}$ ← a sym that exchanges the abelian anyons of spin 3/2.

But $SU(2)_{5-2} = \mathbb{Z}/2 \times \text{Fib}.$

~~∃~~ It would be <u>excellent</u> to know which VOAs have a Fib action.

$$\begin{tikzcd}
 & & Spin(12)_2 \arrow[r, hook] \arrow[dr] & Spin(12)_1^2 \arrow[rr, hook] & & Spin(24)_1 & \\
 & SU(6)_2 \arrow[ur] \arrow[r, hook] \arrow[dr] & SU(6)_1^2 \arrow[r, hook] \arrow[ur] & SU(12)_1 \arrow[urr] & & & \\
Sp(3)_2 \arrow[ur] \arrow[r, hook] & Sp(3)_1^2 \arrow[r, hook] \arrow[ur] & Sp(6)_1 \arrow[ur] & & & Spin(8)_1 \times Spin(16)_1 \arrow[uu, hook] & \\
Sp(1)_2^3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Sp(1)_1^6 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Sp(2)_1^3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & SU(4)_1^3 \arrow[uu, hook] \arrow[rr, hook] \arrow[d, equal] & & Spin(8)_1^3 \arrow[u, hook] \arrow[d, equal, "\text{triality}"] & \\
Spin(3)_1^3 \arrow[r, hook] & Spin(4)_1^3 \arrow[r, hook] & Spin(5)_1^3 \arrow[r, hook] & Spin(6)_1^3 \arrow[r, hook] & Spin(7)_1^3 \arrow[r, hook] & Spin(8)_1^3 \arrow[r, hook] & \cdots \\
Spin(3)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Spin(4)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Spin(5)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Spin(6)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & Spin(7)_3 \arrow[u, hook] \arrow[r, hook] & Spin(8)_3 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal, "\text{triality}"] & \cdots \\
Sp(1)_6 \arrow[r, hook] & Sp(1)_3^2 \arrow[r, hook] & Sp(2)_3 \arrow[r, hook] & SU(4)_3 \arrow[rr, hook] & & Spin(8)_3 &
\end{tikzcd}$$

Map of susy embeddings