The Monstrous Moonshine Anomaly

Cohomology of Groups Seminar,
Perimeter Institute, Nov 27, 2019
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Based on arXiv:1707.08388 and 1810.00463. (joint w/ D. Treumann)

The Fischer–Griess monster $\mathbb{M}$ is the largest sporadic finite simple group. Its order is

$$\#\mathbb{M} = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 13^{3}\cdot 11^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$$
$$\sim 10^{54}.$$

(This isn't *that* large. For instance, $\#SL_{13}(\mathbb{F}_2) > \#\mathbb{M}$.)

It is simple, and so

$$H^1(\mathbb{M};U(1)) = \hom(\mathbb{M},U(1)) = \{1d \text{ reps}\} = *.$$

Furthermore it is known that

$$H^2(\mathbb{M};U(1)) = \{\text{central extensions}\} = *.$$

My goal in this talk is to analyze $H^3(\mathbb{M};U(1))$. This is the home for 't Hooft anomalies of actions of $\mathbb{M}$ on (1+1)d QFTs.

<u>Working prime by prime</u>

Since M is finite, $H^3(M; U(1))$ $\cong H^4(M;\mathbb{Z}) \cong \mathrm{Hom}(H_3 M, U(1))$ is finite, and in fact multiplication by $\# M$ acts trivially. It follows that

$$H^3(M; U(1)) = \bigoplus_{p \mid \# M} \underbrace{H^3(M; U(1))_{(p)}}_{p\text{-part} \,-\, \text{order} = p^k}.$$

Indeed, multiply by $\frac{\# M}{\text{power of } p} \times (\text{normalization})$ to project onto the p^{th} summand.

Furthermore, Sylow theory guarantees the existence of a subgp $S \subseteq M$ s.t. $\# M / \# S$ is coprime to p. Suppose S is such a subgp. Then we have maps

$$H^3(M; U(1)) \underset{\int}{\overset{\text{rest.}}{\rightleftarrows}} H^3(S; U(1))$$

and the composition is multiplication by $\frac{\# M}{\# S}$, hence an iso on p-parts. Thus:

<u>Lemma</u>: For any such S, $\mathrm{rest}|_{H_{(p)}}$ is an injection onto a direct summand.

<u>Example</u>: Let $p \geq 17$ divide M. Then, by studying character tables, there exists a subgp

$$S \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2} \subseteq M.$$

$$\overset{\shortparallel}{\text{**} \subseteq PSL_2(p).}$$

What is the (p)-cohomology of such S?
We can use a spectral sequence:

$$H^{i}\Big(\mathbb{Z}_{\frac{p-1}{2}};\ \underbrace{H^{j}(\mathbb{Z}_p;\ U(1))_{(p)}}_{\substack{\shortparallel \\ \mathbb{Z}_p \text{ in odd degree } 2k-1 \\ 0 \text{ in even degree.}}}\Big) \Rightarrow H^{i+j}(S)_{(p)}.$$

Since p does not divide $\frac{p-1}{2}$, on the E_2 page the only non-zero entries are in the $i=0$ column, and so:

$$H^{j}(S;U(1))_{(p)} = \underbrace{H^{0}(\mathbb{Z}_{(p-1)/2}}_{\text{fixed pts.}};H^{j}(\mathbb{Z}_p;U(1))).$$

So we need to understand this action. It is some power of the action on $H^1=\widehat{\mathbb{Z}_p}$. We learn:

<u>Cor:</u> $H^{\circ}(S;U(1))_{(p)} = \begin{cases}\mathbb{Z}_p & \text{if } \circ=n(p-1)-1 \\ 0 & \text{else.}\end{cases}$

<u>In particular</u>, $H^3(M;U(1))_{(p)}=0$ for $p\geq 17$.

Similar argument works at $p=11$, with

$$S = \text{normalizer of an 11-Sylow} = 11^2:(5\times 2A_5)$$
$$= (\mathbb{Z}_{11}^2 \rtimes(\mathbb{Z}_5\times 2A(t_5)))$$

(Indeed, can use $S=(11\rtimes 5)^2$ plus Künneth.)

Structure of $\mathbb{M}$ at the "small" primes

The primes 2, 3, 5, 7, and 13 are special:

- $p-1$ divides 24
- $\mathbb{M}$ has more than one conjugacy class of cyclic subgp of order p.
- the p-Sylow in $\mathbb{M}$ is nonabelian.

I don't have a complete explanation for why $\mathbb{M}$ treats these primes differently — it is featured in the first paper on Moonshine — but I can give the main ingredient.

To explain it, I need to start with the Leech lattice Λ. This is a rank-24 pos. def. lattice, which means that it is iso to $\mathbb{Z}^{24}$ as an ab. gp, but it has an interesting metric. In fact,

(1) Λ is self-dual (like $\mathbb{Z}^n$ or E_8)
(2) Λ is even (like E_8)
(3) Λ has no roots, i.e. no $\lambda \in \Lambda$ s.t. $\lambda^2 = 2$.

See, any lattice has a root system, and in rank ≤ 24, except for Λ, that root system is full-rank.

In any case, a lattice satisfying (1, 2) determines a holomorphic CFT, which has left-movers in the dual torus $\mathbb{R}^{24}/\Lambda$, and has no right-movers. (If you drop (1), then you will not get a full QFT. If you drop (2), then you will get fermions.) Let's call it V_Λ.

By construction, V_Λ has some manifest symmetries:

- $\mathbb{R}^{24}/\Lambda$ acts by translation.
- $O(\Lambda)$ acts by rotations.

Actually, there is a subtlety — there is an "anomaly", and so the group that acts is a ^usually^ nonsplit extension.

$$\frac{\mathbb{R}^{24}}{\Lambda} \cdot O(\Lambda).$$

Since $O(\Lambda)$ is a finite group, you can understand this extension prime-by-prime. Since $-1 \in Z(O(\Lambda))$ acts by a central character, the extension splits at the odd primes. For the Leech Lattice,

- $O(\Lambda) = Co_0 = 2.\underset{\text{sporadic simple gp.}}{Co_1}$
- $H^2(Co_0, \frac{\mathbb{R}^{24}}{\Lambda}) = H^2(Co_0, 2^{24}) = \mathbb{Z}_2$, and the extension is known not to split.

~~There are also typically more symmetries~~
There would be more symmetries if Λ had roots.
Incidentally,

$$\# Co_0 = 2^{22}.3^9.5^4.7^2.11.13.23 \sim 10^{18}.$$

In any case, take $g \in Co_0$. It may have nonequiv. lifts to V_Λ, because of the extension. But

Let's assume that g acts on $\mathbb{R}^{24}$ w/ no fixed pts, and $o(g) = p$ is prime.
Since Λ is defined over $\mathbb{Z}$, the char poly of g must factor as $(x-1)^{\#\text{fixed pts}} \cdot (x^{p-1} + \dots + x + 1)^{\text{rest}}$, and so we must have $p-1$ divides 24.
It turns out that Co_0 has a conj. class of g for each such p, called

"2A, 3A, 5A, 7A, 13A".

It furthermore turns out that these $\mathbb{Z}_{pA}$ all act nonanomalously on V_Λ, and so we can gauge the action

$$V_\Lambda // \mathbb{Z}_{pA}.$$

The gauging procedure screens the free boson modes (generators of translation) in V_Λ, and when you calculate the spins of the twisted sectors, you find:

$(V_\Lambda // \mathbb{Z}_{pA})$ has no fields of spin 1.

In fact, as conjectured by [FLM] and proven recently [Abe, Lam, Yamada],

$$V_\Lambda // \mathbb{Z}_{pA} \cong V^\natural$$

is the Moonshine CFT.

What symmetries does this make manifest?
The choice of $\mathbb{Z}_{pA}$ breaks all but ~~the~~ its normalizer in $\mathrm{Aut}(V_\Lambda)$. What survives are:

- The $\mathbb{Z}_{pA}$-fixed elts of $\frac{\mathbb{R}^{24}}{\Lambda}$ is a subgp of shape p^d, $d = \frac{24}{p-1}$.
- $\mathbb{Z}_{pA}$'s normalizer in Co_0, which has shape ~~$\mathbb{Z}_{pA}$~~ $\mathbb{Z}_{pA} \cdot J$ with

p	J		splits?
2	Co_1	← normalizing but not centralizing.	×
3	$2Suz.2$		×
5	$2J_2.4$		✓
7	$3 \times 2S_7$		✓
13	$3 \times 4S_4$		✓

After gauging, you ~~lose~~ lose the $\mathbb{Z}_{pA}$ symmetry, but pick up a dual symmetry ~~to~~ "$\widehat{\mathbb{Z}_{pA}}$".
It turns out:

- The p^d gp extends to an extraspecial p^{1+d} (This has an explanation having to do with anomalies for the $\frac{\mathbb{R}^{24}}{\Lambda}$ action).

I don't know why!

- "$\widehat{\mathbb{Z}_{pA}}$" is of class pB in $\mathbb{M}$.
- The resulting gp
$$p^{1+d}.J \subseteq \mathbb{M}$$
contains the p-Sylow.

Ex: At $p=13$, we have

$$S = 13^{1+2}:(3\times 4S_4).$$

Run the Serre s.s.:

$$H^\bullet(\underbrace{3\times 4S_4}_{\text{no }13\text{s}}; H^\bullet(13^{1+2}))_{(13)} \Rightarrow H^\bullet(S; U(1))_{(13)} \cong H^\bullet(\mathcal{M})_{(13)}$$

so the E_2 page is $H^0(3\times 4S_4; H^3(13^{1+2}; U(1)))$.

What is 13^{1+2}? ~~[illegible]~~ It is a Heisenberg gp $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1\end{pmatrix} \subseteq SL_3(13)$. What is its coh? Another SS:

$$\overset{E_2}{\overset{\parallel}{H^\bullet}}(13^2, H^\bullet(13)) \Rightarrow H^\bullet(13^{1+2})$$

$$\begin{array}{c|cccc}
3 & 13 & & & \\
2 & 0 & 0 & 0 & 0 \\
1 & 13 & 13^2 & 13^3 & 13^4 \\
0 & U(1) & 13^2 & 13 & 13^3 \\
\hline
 & & \ast & \overset{\parallel}{Alt^2(13^2)} & \overset{\parallel}{Alt^3(13^2)\oplus Sym^2(13^2)}
\end{array}$$

Find out:

$H^1 = 13^2 = E^*$

$H^2 = Alt^2(E^*)/\text{antisym. form} = 0$

$H^3 = Sym^2(H^2 \otimes \text{line})$

So $H^3(13^{1+2}) = Sym^2(E^*) \simeq 13^3$.

But action of $J = 3\times 4S_4$ on ~~13~~ $E = 13^2$ has a nontrivial central character, and so still acts nontriv on $Sym^2(E^*)$, so $H^0(3\times 4S_4; H^3(13^{1+2})) = 0$.

So $H^3(M; U(1))_{(13)} = 0$.

Let's repeat. For $p=7$, we have: $E = 7^4$,

$H^3(7^{1+4}; U(1)) = Sym^2(E^*) \cdot \left(\underset{\omega}{\underline{Alt^2(E^*)}} \otimes \mathfrak{g}(\omega)\right)$
$H^2 \qquad\qquad = \underline{Alt^2(E^*)}\ \cancel{\omega}$
$H^1 = E^*$.

Again: centre of $J = 3 \times 2S_7$ acts ~~sufficiently~~ by untriv. char. on all of these. Also $H^3(J; U(1))_{(7)} = 0$.
So E_2 page vanishes.

~~A~~ Let's repeat for $p=5$. Same argument. Need to use

Lemma: $H^3(2J_2.4; U(1))_{(5)} = 0$.
Pf: Computer ~~calc, too big brute force.~~
$\Rightarrow H^3(2J_2; U(1))_{(5)} = 5$.
The outer 4 action is nontrivial
(~~#~~ calculated it w/ Treumann).
So SS ~~⇒~~: $H^3(2J_2.4) \Leftarrow H^0(4, H^3(2J_2)) = 0$.

~~For~~

For the prime $p=3$, you try to do the same. On the E_2 page, you find, with $J = 2Suz2$, $E = 3^{12}$,

$$\begin{array}{c|llll}
3 & H^0(J; Sym^2 E^* \oplus \boxed{Alt^2 E^*/\omega}) & & & \\
2 & & H^1(J; Alt^2 E^*/\omega) & & \\
1 & & & \boxed{H^2(J; E^*)} & \\
0 & & & & H^3(J; U(1)) \\
\hline
\end{array}$$

Actually, by thinking about the centre of J, the boxed parts are zero. Furthermore:

- E^* is not sym. self dual, so $H^0(J; Sym^2 E^*) = 0$.
- $H^3(2Suz.2; U(1))_{(3)} = 0$.

(In fact, $H^3(2Suz) = \mathbb{Z}_8$. This was a hard by-hand calculation, of the same flavour as I am explaining, and required the Lie grp E_6)

So: $H^3(M; 3) \leq H^1(J; Alt^2 E^*/\omega)$

$\leq H^1(3^5 : (M_{11} \times 2); \checkmark)$ ← contains 3-Sylow in J

$= \mathbb{Z}_3$ by computer.

Is $H^3(M; 3)$ nontrivial? Yes. I will explain why, because it is a nice technique.

See, look at $\mathbb{Z}_{3B} = \widehat{\mathbb{Z}_{3A}} \le M$. Its ~~centralizer~~ centralizer, we said already, is

$$3^{1+12} \cdot 2Suz. \curvearrowright V^\natural$$

We can (un)gauge it to get back to V_Λ. It turns into

$$3^{12} : 6Suz \curvearrowright V_\Lambda.$$

Why? Let's look at the s.s. for

$$3^{1+12} \cdot 2Suz = 3G$$

On the E_2 page:

$\hat{3} = \mathrm{hom}(\mathbb{Z}_{3B}, U(1))$

$$\begin{array}{c|ccccc}
3 & Sym^2(\hat{3}) & & & & \\
2 & 0 & 0 & 0 & 0 & \\
1 & \hat{3} & H^1(G;\hat{3}) & H^2(G;\hat{3}) & H^3(G;\hat{3}) & \\
0 & U(1) & H^1(G) & H^2(G) & H^3(G) & H^4(G) \\
\hline
\end{array}$$

The d_2 differential we already understand: it is

$$\alpha \mapsto \langle \alpha \cup \kappa \rangle$$

where $\kappa \in H^2(G; 3)$ classifies the extension.

~~Further~~ Write $\omega^\natural$ for the anomaly of $M \otimes V^\natural$.

Then:

$$\omega^\natural|_3 = 0.$$

~~So~~ And

$$\omega^\natural|_{3G} \in H^3(3G) =$$

$$\underbrace{(\text{sub of } H^3(3))}_{\text{nothing here.}} \cdot \left(\text{ker}: d_2 : H^2(G; \hat{3}) \to H^4(G, \ldots\right.$$
$$\cdot\ (\text{coker}: d_2 : H^1 \to H^3).$$

So we have:

$$\omega^\natural|_{3G} \in (\text{ker}) \cdot (\text{coker})$$
$$= \text{"}(\alpha, \beta)\text{"}.$$

~~Warning: extension can split!~~

Warning: This extension might be nontrivial.

So α is a well-defined thing, i.e. $d\alpha = 0$

but

$$\boxed{d\beta = \langle \alpha \cup \kappa \rangle}$$

Thm: On the dual side, the rôles of α and κ are exchanged.

In particular, α = extension data for

$$3^{12} : 6Suz = 3 \cdot (3^{12} : 2Suz).$$

This is non-zero! Cor: $H^3(M; U(1))_{(3)} \neq 0$.

The $p=2$ case

This is the hardest. We have an upper bound $H^\bullet(\mathbb{M})_{(2)} \le H^\bullet(2^{1+24}\cdot Co_1)$. Let's run the s.s.. Write $E=2^{24}$. Then the E_2 page is:

	$H^\bullet(2^{1+24})$	dim
3	complicated. exp $=4$.	$= 2^{2300}-2$
2	$\mathrm{Alt}^2(E^*)/\text{line}$	$= 2^{275}$
1	E^*	$= 2^{24}$
0	$U(1)$	

$$\begin{array}{c|llll}
3 & H^0(Co_1, 2^{2300}-2) & & & \\
2 & H^0(Co_1, 2^{275}) & H^1(Co_1, 2^{275}) & & \\
1 & H^0(Co_1, 2^{24}) & H^1(Co_1, 2^{24}) & H^2(Co_1, 2^{24}) & \\
0 & U(1) & H^1(Co_1) & H^2(Co_1) & H^3(Co_1) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

$$= \begin{array}{|llll}
\le 4 & & & \\
\bullet & 2 & & \\
\bullet & \bullet & 2 & \\
U(1) & \bullet & 2 & \boxed{4} \\
\hline
\end{array}$$

≤ 4 ← LES analysis.

2 ← me. Compute brute force.

2 ← D. Holt. Extension $2^{24}.Co_1$

$\boxed{4}$ ← Thm (JF, Treu)

2 ↑ double cover $2.Co_1 = Co_0$

and so an E_∞ page $\le$ this.
This gives some upper bounds. But we need a lower bound.

Look at $3^{1+12}.2Suz.2 \subseteq \mathbb{M}$. Since $3^{12}:2Suz$ splits, we can find $6Suz \subseteq \mathbb{M}$. Central 2 is of class $2B \in \mathbb{M}$, so $6Suz \subseteq 2^{1+24}\cdot Co_1$, mapping over $3Suz \subseteq Co_1$. Now, inside $\cancel{2}A_7$

There is a $D_8 \subseteq 3Suz \subseteq Co_1$,
lifting to
$$2D_8 \subseteq 6Suz \subseteq Co_0.$$
Meng explained how to calculate $H^\bullet(\cancel{2}D_8)$
What about $2D_8$?

Turns out:
$$\begin{array}{c} 2D_8 \subseteq SU(2) \simeq S^3. \\ \downarrow \\ D_8 \subseteq \cancel{SO(3)}\ O(2) \leq SO(3) \end{array}$$

Consider the S.S. for $S^3 \to S^3/2D_8$
$$\downarrow$$
$$B\,2D_8.$$

It says:
$$H^\bullet_{gp}(2D_8;\, H^\bullet_{space}\, S^3) \Rightarrow H^\bullet_{spc}(\underbrace{S^3/2D_8}_{\text{a 3-manifold.}})$$

when you run it, you learn:
$$H^3_{gp}(2D_8\cancel{\ };\, U(1)) = H^4_{gp}(2D_8;\mathbb{Z}) = \mathbb{Z}/_{\#2D_8}$$
$$= \mathbb{Z}_{16}.$$

OTOH, under the orbifold,
$$2D_8 \subseteq 2^{1+24}\cdot Co_1 \longrightarrow 2D_8 \subseteq 2^{24}\cdot Co_1,$$

and so $\omega^A|_{2D_8}$ must implement this, i.e.
it must have nontrivial class in $H^2(D_8, \hat{2})$,

$$\begin{array}{cccc} 2 & & & \\ & 0 & & \\ & & 2 & \\ & & & \text{exp}=4 \end{array}$$

and that forces:

(*) $\omega^{\natural}|_{2D_8}$ has exact order 8.

On the other hand, by nontrivial calculation, ~~there does not exist a class in $H^3(G\,Suz)$~~ which

$$H^3(G\,Suz) \to H^3(2D_8)$$

has image the $\mathbb{Z}_8 \subseteq \mathbb{Z}_{16}$ (even elts).

So:

(**) $\omega^{\natural}$ is not divis. by 2.

Cor 1: $\omega^{\natural}$ generates a direct summand of $H^3(M)_{(2)}$.

Cor 2: on the E_∞ page, $E_\infty^{03} = 2$:

$$\begin{array}{cccc} 2 & & & \\ & \leq 2 & & \\ & & \leq 2 & \\ & & & 4. \end{array}$$

Cor 3: $8\omega^{\natural}|_{2^{1+24}\cdot Co_1}$ is pulled back from Co_1

But: By thm w/ Treumann, $(2)D_8 \subseteq Co_1$ detects $H^{\bullet}_{(2)}$.

and $8\omega^H\big|_{2D_8} = 0$, and so:

Cor 4: ω^H has exact order $8 \times (\text{odd})$.

■

Summary: $H^3(M; U(1)) = \mathbb{Z}_{24} \oplus (\leq 4)$