The Stokes groupoids of Gualtieri, Li, and Pym

talk by Theo Johnson-Freyd, 4 May 2015,
Northwestern University Math 448: Topics in Geometry & Topology.

This talk is based on the paper
Gualtieri, Li, and Pym, "The Stokes groupoids,"
arXiv: 1305.7288 (2013).
No results herein are due to the speaker.

The topic of the paper is linear ordinary differential equations with singularities, possibly irregular. Remember that if you have some diff eq like

$$f'' + af' + bf = 0, \quad a, b \in \mathcal{O}$$

you can make it 1st order by studying the equations

$$\left(d + \begin{pmatrix} 0 & -1 \\ b & a \end{pmatrix}\right)\begin{pmatrix} f \\ f' \end{pmatrix}.$$

So I will talk about 1st order ODEs.

Fix a curve X. A 1st order ODE is a vector bundle $\mathcal{E} \downarrow X$ and a connection, which recall looks locally like $d + A(z)dz$. Let D be an effective divisor on X. What I really want to talk about are meromorphic connections with poles bounded by D. These look like $d + A(z)dz$ except that if we're in coordinates centered at $p \in D$ with multiplicity k, then $A(z) = \frac{a}{z^k} + \cdots$.

But first, let me analyze the non-singular case, to set the vocabulary. A connection is a <u>representation</u> of the tangent bundle — you can take this as a definition of $\mathrm{Rep}(T_X)$. For me "representations" will always be on finite-dim vector bundles. Recall that a <u>Lie groupoid</u> is a space G with ~~maps~~ surjective submersions

$$\begin{array}{c} G \\ s\downarrow\ \downarrow t \\ X \end{array}$$

and an identity bisection $i: X \to G$, $si = ti = id_X$, and an associative multiplication $\begin{array}{c} G \times^{X}_{t,s} G \\ \downarrow s \qquad \downarrow t \\ X \end{array} \to \begin{array}{c} G \\ s\downarrow\ \downarrow t \\ X \end{array}$

and a grouplike axiom that I won't write (it's property, not data). A <u>representation</u> of G on a vector bundle E is an isomorphism

$$\Psi: t^*E \xrightarrow{\sim} s^*E, \text{ i.e. } \begin{array}{c} G \times^{X}_{t} E \\ \downarrow s \\ X \end{array} \rightrightarrows \begin{array}{c} E \\ \downarrow \\ X \end{array}.$$

which is associative.

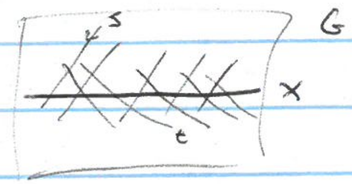
For example, X has a <u>pair groupoid</u> $\mathrm{Par}(X) = \begin{array}{c} X \times X \\ \downarrow\downarrow \\ X \end{array}$ with s, t = the two projections. It also has a <u>fundamental groupoid</u> $\pi_1 X$, given by paths mod (homotopy rel boundary). Each source fiber is a simply connected cover of X: $s^{-1}(x) \xrightarrow{t} X$.

A representation of $\mathrm{Par}(X)$ is a trivialized bundle. The <u>Riemann-Hilbert correspondence</u> says

$$\mathrm{Rep}(\Pi_1 X) \xrightarrow{\text{"differentiate"}} \mathrm{Rep}(T_X)$$

is an equivalence of categories.

Given a Lie groupoid, you can study an infinitesimal neighborhood of $i(X) \subseteq G$.



The first-order nbhd can be identified with

$A = i^*(\ker(ds)) = $ "Lie(G)"

← the "source-vertical" tangent bundle.

Then the residual data of t is $\tau = dt : A \to T_X$, a map of sheaves. The residual data of the multiplication is that A is valued in Lie algebras, and τ is a Lie alg map. The Lie bracket on sections of A is not $\mathcal{O}_X$ linear, but rather satisfies a Leibniz rule. ~~Note that~~

Then the RH map is a special case of

<u>Lie II theorem (oid version)</u>: Suppose that $A = \mathrm{Lie}(G)$ and that the source fibers of G are connected simply connected. Then

$$\mathrm{Rep}(G) \xrightarrow{\text{differentiate}} \mathrm{Rep}(A)$$

is an equiv.

<u>Pf:</u> Existence & uniqueness of solns to ODEs.

Warning: For groups, the "Lie III theorem" says that every Lie alg is Lie(gp). This fails for oids. In fact:

A representation of A is a "connection" ~~$A \otimes E \to E$~~ $E \to E \otimes A^\vee$ ~~with some properties~~.

Lie III counterclaims:

- ∃ Lie algebroid A s.t. ∄ G with $A = \mathrm{Lie}(G)$.
- if ∃ G with $A = \mathrm{Lie}(G)$, then ∃ $\tilde{G}$ with $A = \mathrm{Lie}(\tilde{G})$ and source-simply-connected. But $\tilde{G}$ might not be Hausdorff.

OK, back to ODEs. Recall we want to study (the representation theory of)

$$\nabla = \partial + A(z)\,\partial z$$

where $A(z)$ is meromorphic with poles bounded by the divisor $D \subset X$. Such a connection cannot pair with all vector fields. But it can pair with vector fields that vanish along the divisor!

~~$\nabla_{z\partial_z}$~~ - if $\nabla = \partial + \frac{1}{z}\partial z$, then $\nabla_{z\partial_z} = z\partial_z + 1$.

Definition: The twisted tangent bundle $T_X(-D)$ is the subsheaf of T_X whose sections vanish along D.

Lemma: $T_X(-D)$ is a locally free sheaf of rank $= \dim X = 1$, The inclusion $T_X(-D) \hookrightarrow T_X$ makes it into a Lie algebroid.

~~So a meromorphic connection ... is a rep of T_X~~

So we want to study $\mathrm{Rep}(T_X(-D))$.

The corollary to Lie II says:

Suppose ∃ a groupoid "$\Pi_1 X(-D)$" which is source-simply-connected with $\mathrm{Lie}(\Pi_1 X(-D)) = T_X(-D)$. Then $\Pi_1 X(-D)$ is the universal domain of definition for solutions to ODEs with poles bounded by D.

~~Ker: every such connection defines a holomorphic bundle on $\Pi_1 X(-D)$~~

The main point of the paper is to construct $\Pi_1 X(-D)$.

Blow-up of groupoids ~~along~~ div

The following can be generalized to higher (co) dimension.

Let $G \rightrightarrows X$ a groupoid and $p \in X$. Then

$$G(-p) = \begin{cases} \cdot \text{ blow up } G \text{ along } i(p) \\ \cdot \text{ remove } s^{-1}(p) \text{ and } t^{-1}(p). \end{cases}$$

If ~~D is~~ $D = k_1 p_1 + \dots + k_n p_n$ is an effective divisor, then $G(-D) = G\underbrace{(-p_1)(-p_1)\dots(-p_1)}_{k_1 \text{ times}}(-p_2)\dots$

Thm: $\mathrm{Lie}(G(-D)) = \mathrm{Lie}(G)(-D)$.

Corollary: $\Pi_1 X(-D)$ always exists, although it might not be Hausdorff.

Defn: ~~$\Pi_1 X(-D)$ is~~ Π_1 –

Since Π_1 can fail to be algebraic even when Hausdorff, let's start with $\mathrm{Par}(X)(-D)$:

<u>Examples</u>:

- $\mathrm{Par}(\mathbb{A}^1)(-k\cdot\{0\}) \equiv \mathbb{A}^2_{z,u} \setminus \{1 + u z^{k-1} = 0\}$

$$s \downarrow \quad \downarrow t$$
$$\mathbb{A}^1$$

$s(z,u) = z. \quad t(z,u) = (1 + u z^{k-1}) z$

$(z_2, u_2)(z_1, u_1) = (z_1, u_2(1 + u_1 z^{k-1})^k + u_1).$

~~In particular, $\mathrm{Par}(\mathbb{A}^1)(-\{0\}) =$ ... $\mathbb{A}^1 // \mathbb{C}^\times_{(u+1)}$~~

In particular, $\mathrm{Par}(\mathbb{A}^1)(-0) = \mathbb{A}^1_z // \mathbb{C}^\times_{(u+1)}$.

- $\mathrm{Sto}_k := \Pi_1 \mathbb{A}^1(-k\cdot 0) = \mathbb{A}^1_z \times \mathbb{C}_u$

$$s \downarrow \quad \downarrow t$$
$$\mathbb{A}^1$$

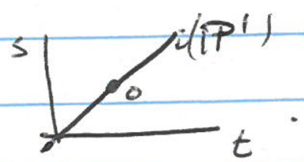
$s(z,u) = z, \quad t(z,u) = \exp(u z^{k-1}) z$

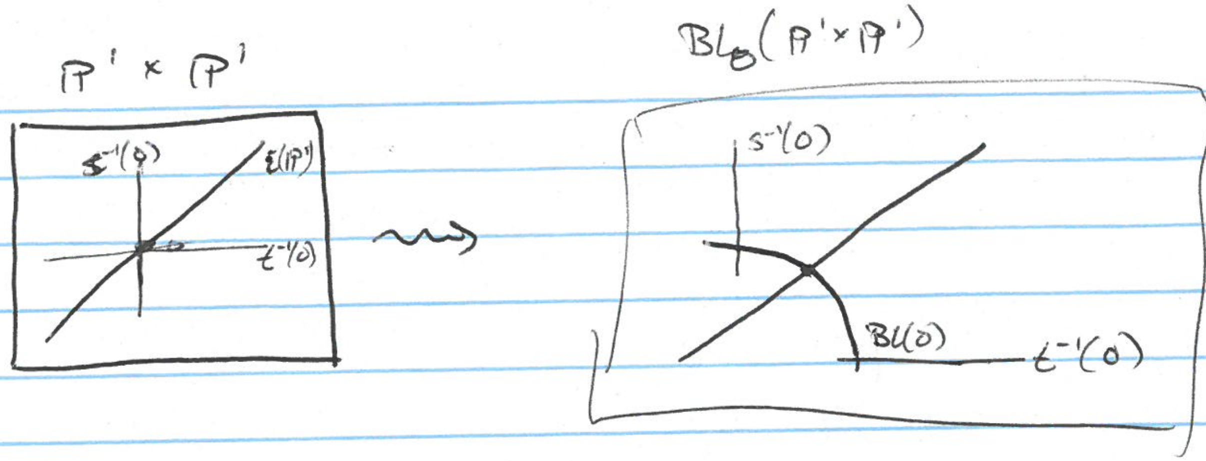
$(z_2, u_2)(z_1, u_1) = (z_1, \; u_2 \exp((k-1) u_1 z_1^{k-1}) + u_1)$

In particular, $\mathrm{Sto}_1 = \mathbb{A}^1_z // \mathbb{C}$ via exponential action.

- $\mathrm{Par}(\mathbb{P}^1)(-0)$:

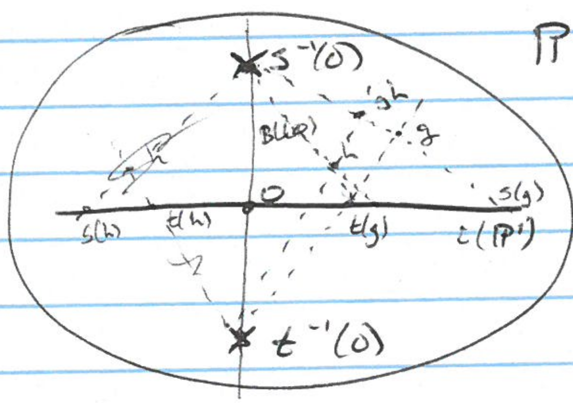
well, $\mathrm{Par}(\mathbb{P}^1) = \mathbb{P}^1 \times \mathbb{P}^1$ ~~$\mathbb{A}$~~ $\mathbb{P}^1$





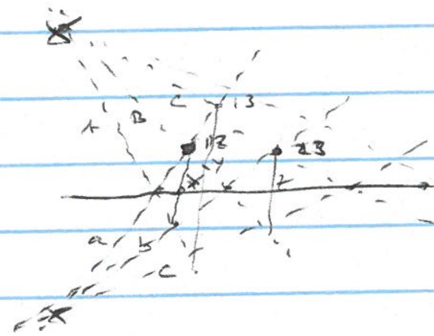
now remove $s^{-1}(0)$, $t^{-1}(0)$.

Clearer: Blow down $s^{-1}(0)$, $t^{-1}(0)$ before removing:



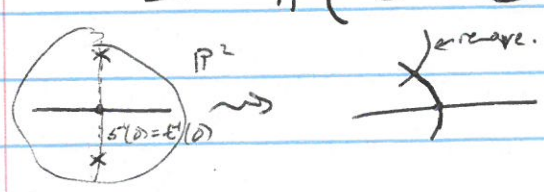
$\mathbb{P}^2$ = blowdown.

Associativity = a theorem of Pappus.



- $\mathrm{Par}(\mathbb{P}^1)(-(0+\infty)) = \mathbb{P}^1 // \mathbb{C}^*$ multiplicative.
  $\Pi_1 \mathbb{P}^1(-(0+\infty)) = \mathbb{P} // \mathbb{C}$ exponential action.

- $\mathrm{Par}(\mathbb{P}^1)(-2\cdot 0) = \Pi_1 \mathbb{P}^1(-2\cdot 0)$
  $= \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \setminus \pi^{-1}(p_\infty)$ where $p_\infty \in E$
  E is the exceptional divisor.



Thm: For $k \geq 2$, $\text{Par}(\mathbb{P}^1)(-k \cdot p) = \text{Tot}(\mathcal{T}_{\mathbb{P}^1}(-k \cdot p))$ compatible with source/bundle maps.

Thm: For all cases except ~~($\mathbb{P}^1, p$)~~, $(\mathbb{P}^1, p)$,
$\mathcal{T}_{\cdot}$ $\mathcal{T}_1 X(-D)$ is Hausdorff.
~~$\mathcal{T}_1$(~~

The last part of the paper has to do with lifting formal solutions to actual solutions. Consider, as an example,

$$\nabla = d + \frac{a}{z^2} + \frac{b}{z} + (\text{regular}),$$

a rep of $\mathcal{T}_{\mathbb{A}^1}(-2 \cdot 0)$.

As Peng explained last time, by a gauge transformation of the form $P + zQ$, we can assume a, b simultaneously diagonalized. Then the usual approach to solving ODEs goes:

(0) if (regular) $= 0$, then

$$df_i + \left(\frac{a_i}{z^2} + \frac{b_i}{z}\right) f = 0 \rightsquigarrow \frac{df}{f} = -a_i z^{-2} - b_i z^{-1},$$

$$\rightsquigarrow \log f_i = a_i z^{-1} - b_i \log z$$

$$\rightsquigarrow f_i = e^{a_i z^{-1}} z^{b_i}.$$

(1) So as an Ansatz, assume

$$f_i = e^{a_i/z} z^{b_i} \cdot c_i(z)$$

where $c_i(z) = \sum c_i^{(n)} \frac{z^n}{n!} \in \mathbb{C}[[z]]$.

(2) Write down a "transfer equation", which finds $c_i^{(n)}$ as a function of earlier terms.

You end up with a "formal solution". But in almost all cases, $\sum C^{(n)} \frac{z^n}{n!}$ has zero radius of convergence! How to interpret this solution?

Here's the answer. Vector $\vec{f}$ is ~~an Iso~~ a formal isomorphism between two reps of $\pi_1/A^1(-2\cdot 0)$, the one we want (call it Σ_1) and the diagonal one (Σ_0). Both correspond to ~~rep~~ (holomorphic) reps of $\pi_1/A^1(-2\cdot 0)$. So $\vec{f}$ is the Taylor expansion of a holomorphic function on $\pi_1/A^2(-2\cdot 0)$, or, rather, its change-of-variables along s,t is.

In particular, the first few terms of the solution to Σ_1, when written on $\pi_1/A^2(-2,0)$, depend only on the first few terms of $\vec{f}$. So this gives you analytic control over Σ_1.