Holomorphic SCFTs with small index

GMU, 13 Nov 2020
Theo Johnson-Freyd

based on arXiv:1811.00589 jt w/ Davide Gaiotto.
these slides are available at http://categorified.net/Nov13slides.pdf

Challenge problem:

field w/ 3 elts $= \{0, 1, -1\}$

Suppose $C \subseteq \mathbb{F}_3^n$ s.t.

$$C = C^\perp := \{x \in \mathbb{F}_3^n : x \cdot y = 0 \ \forall y \in C\}.$$

self-dual, aka error-correcting, ternary code.

Define

$$\mathrm{Index}(C) := \sum_{x \in C} \underbrace{\prod_{i=1}^{n} x_i}_{\in \{0,1,-1\}} \in \mathbb{Z}$$

(A) Prove that $\mathrm{Index}(C) \in 24\mathbb{Z}$.
Trivial unless $n = 12k$. I can do k even.

(B) Produce C's with $\mathrm{Index}(C) = 24$.
For all $n = 12k$. I can do $k \le 5$.

Big Question:

Understand {QFTs}. (a) Construct elements in this space. (b) Construct functions on this space.

My focus: 2D (= 1+1 D).

{2D QFTs}	mathematically ill-defined	infinite-dim deformations.
$\cup\vert$		
{2D CFTs}	mathematical defn is on the horizon	finite-dim deformations
$\cup\vert$		
{2D holomorphic CFTs}	mathematically well-defined.	isolated, rigid, special.

Idea of defn: A QFT is holomorphic if all local operators depend only holomorphically on their locations.

Big Question: (minimal) supersymmetry, aka "$N=(1,0)$".

Understand $\{SQFTs\}$. (a) Construct elements in this space. (b) Construct functions on this space.

My focus: 2D (= 1+1 D).

$\{$2D SQFTs$\}$ — mathematically ill-defined — infinite-dim deformations.

$\cup|$

$\{$2D SCFTs$\}$ — mathematical defn is on the horizon — finite-dim deformations

$\cup|$

$\{$2D holomorphic SCFTs$\}$ — mathematically well-defined. — isolated, rigid, special.

Warm-up: SQM = K

Warm-up 1 (= 0 + 1)D QFT = QM.

$\{1D\ QFTs\} := \{ (\mathcal{H}, \hat{H})$ where $\mathcal{H}$ is a Hilbert space and $\hat{H}$ is a self-adjoint operator

the Hamiltonian

s.t. $\mathrm{tr}_{\mathcal{H}} \exp(-t\hat{H}) =: Z(t)$ converges absolutely $\forall\, t > 0. \}$

the Wick-rotated partition function.

Topologized via strong convergence of the resolvent.

$$\boxed{\begin{array}{l}\text{Exercise: } \{1D\ QFTs\} \text{ is contractible.} \\ \text{Idea: push all e-values to } \infty.\end{array}}$$

$\cup\!\!|$

$\{1D\ CFTs\}$

$\|$

$\{1D\ TFTs\} = \{(\mathcal{H}, \hat{H}) : \hat{H} = 0.$ Forces $\dim \mathcal{H} < \infty. \}$

$\hookrightarrow \{1D\ TFTs\} = \coprod_{n \in \mathbb{N}} BU(n).$ Important in the study of "gapped phases".

Warm-up: 1 (= 0 + 1) D SQFT = SQM.

(minimal susy = "$\mathcal{N}=1$")

$$\{1D \text{ SQFTs}\} := \{(\mathcal{H}, \hat{G}, \hat{H})\} \quad \text{where}$$

$\hat{G}$: the "supersymmetry operator"

$\mathcal{H} = \mathcal{H}_{ev} \oplus \mathcal{H}_{odd}$ is a super Hilbert space,

Set $(-1)^F := \quad +1 \quad\quad -1$.

$\hat{G}$ is an odd self-adjoint operator

i.e. $\hat{G}(-1)^F = -(-1)^F \hat{G}$

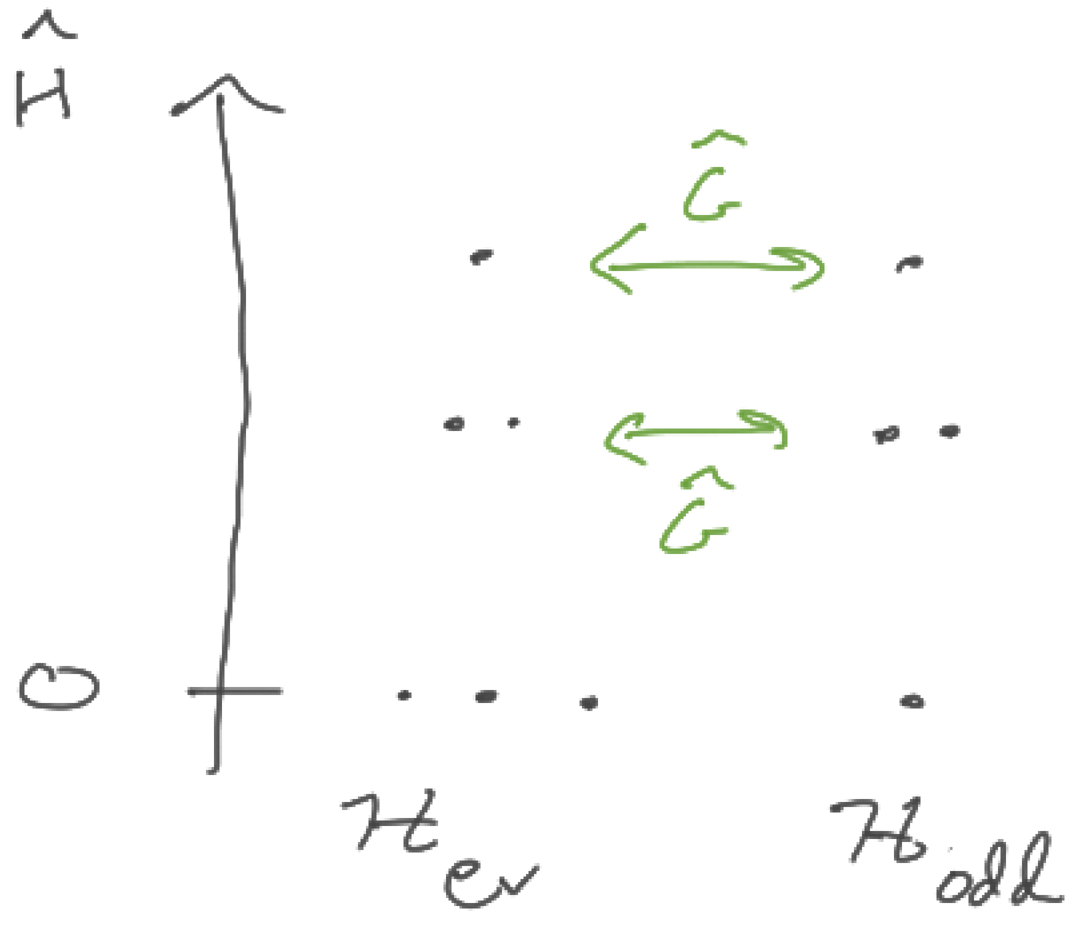
$\hat{H} = \hat{G}^2$, s.t.

$\chi(t) := \mathrm{tr}_{\mathcal{H}}\left((-1)^F \exp(-t\hat{H})\right)$ converges absolutely $\forall t > 0$.

$\hookrightarrow$ i.e. $\exp(-t\hat{H}) = \exp(-t\hat{G}^2)$ is trace class.

$$\chi(t) := \operatorname{tr}\left((-1)^F \exp(-t\hat{H})\right).$$

Absolute convergence $\Rightarrow$ spectrum of $\hat{H}$ is discrete, finite multiplicity



$\hat{H} = \hat{G}^2 \Rightarrow$ non-zero e-values come in pairs

$\Rightarrow$ cancel in χ.

$\Rightarrow \chi$ is a constant. It counts (with sign) the ground states $\hat{H} = 0$.

In physics, χ is called the Witten index of $(\mathcal{H}, \hat{G})$.

$$\begin{tikzcd} \{\text{1D SQFTs}\} \arrow[r, "\chi"] & \mathbb{Z} \\ \{\text{1D STFTs}\} \arrow[u, phantom, "\cup" description] \arrow[ur] & \end{tikzcd}$$

is a <u>deformation invariant</u>.

Moreover, χ is "complete" in the sense that it is an iso on π_0 *.

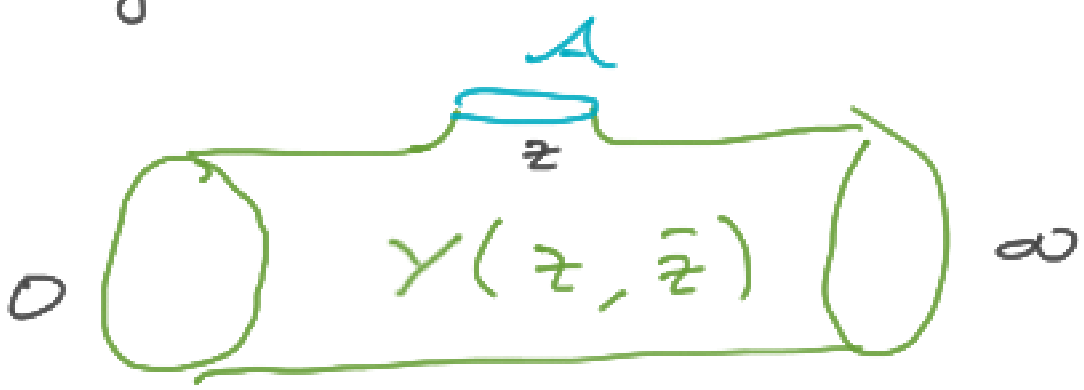
In fact, $\{\text{1D SQFTs}\} \simeq KU$.

* This is true for $\{\text{1D SQFTs}\}$ w/ the "strong conv. of the resolvent" topology, and <u>not</u> for $\{\text{1D STFTs}\}$.

Holomorphic (S)CFTs

What is a holomorphic CFT?

Any CFT has a Hilbert space $\mathcal{A}$ of states on S^1.



$$= \mathbb{CP}^1 \setminus \{0, \infty, z\}$$

State-operator correspondence:

Y gives an isomorphism $\mathcal{A} \cong$ algebra of operators.

$\rightsquigarrow Y(z,\bar{z}) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

Holomorphic: $\bar{\partial} Y = 0$. i.e. $Y(z)$ is holo. i.e. $\mathcal{A}$ is a VOA.

Not every VOA arises!

For $\mathcal{A}$ to be a full alg. of operators,

$$\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Vec}.$$

Defn: A holo CFT is a unitary VOA w/ $\mathrm{Rep}(\mathcal{A}) \cong \mathrm{Vec}$.

Compare QM case: operators $= B(\mathcal{H})$, $\mathrm{mod}(B(\mathcal{H})) \cong \mathrm{Vec}$.

I won't review the full defn of VOA, just review some features.

① Rotation of S^1 $\rightsquigarrow$ $\mathbb{N}$-grading $A = \bigoplus_{n \geq 0} A_n$ $L_0 :=$ op w/ e-value n on A_n.

② OPE: given $a, b \in A$,

$a(z)b(0) =$ (meromorphic in z) $=$ $\underbrace{(\text{polar part})}$ + smooth.

this is the analog of the commutator in an assoc. alg.

③ Example: stress-energy tensor $T \in A$ solves

$$T(z)T(0) \sim \frac{c}{z^4} + \frac{2T}{z^2} + \frac{\partial T}{z}$$

$8\mathbb{Z}$ since holomorphic.

$\overset{\cup\!\cup}{c}$ is the central charge aka grav. anomaly.

④ The Hamiltonian is $L_0 - \frac{c}{24}$.

⑤ The character is $\chi(q) = \mathrm{tr}_A\left(q^{L_0 - \frac{c}{24}}\right) = \sum_n (\dim A_n)\, q^{n - \frac{c}{24}}$.

Zhu modularity: $\chi(q)$ is a (level-1) modular fn w/ multiplier.

What is a holomorphic SCFT?

Two super Hilbert spaces: $\mathcal{A}_{NS} \longleftrightarrow S^1$ w/ bounding spin str.

$\mathcal{A}_R \longleftrightarrow S^1$ w/ nonbounding spin str.

super

So $\mathcal{A}_{NS} =: \mathcal{A} = \mathcal{A}_{ev} \oplus \mathcal{A}_{odd}$

$\mathcal{A}_R = (\mathcal{A}_R)_{ev} \oplus (\mathcal{A}_R)_{odd}$

$L_0 \in$	$\mathbb{N}$	$\mathbb{N}+\frac{1}{2}$	$\frac{1}{2}\mathbb{N}+\frac{c}{24}$.
	$\mathcal{A}_{ev}$	$\mathcal{A}_{odd}$	$\mathcal{A}_R$

$\mathcal{A}$ is a super VOA. $\mathcal{A}_0$ is an ordinary VOA. $\mathcal{A}_R$ is an $\mathcal{A}_0$-mod.

Holomorphic: $SRep(\mathcal{A}) = SVec$. Equiv: $Rep(\mathcal{A}_0) = \{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_R\}$.

Supersymmetry operator: $G \in (\mathcal{A}_{odd})_{3/2}$ solving $G(z)G(0) \sim \frac{\frac{3}{2}c}{z^3} + \frac{T}{z}$.

Action of G on $\mathcal{A}_{NS}$ is $G(z) = \sum_{n\in\mathbb{Z}} \hat{G}_{n+\frac{1}{2}} z^{n+\frac{1}{2}}$

true for any odd operator.

Action of G on $\mathcal{A}_R$ is $G(z) = \sum_{n\in\mathbb{Z}} \hat{G}_n z^n$.

On $\mathcal{A}_R$, $\hat{G}_0^2 = \hat{H} = L_0 - \frac{c}{24}$. So $\chi := Tr_{\mathcal{A}_R}\left((-1)^F q^{L_0-\frac{c}{24}}\right) \in \mathbb{Z}$.

Engineering SCFTs

Any holomorphic SCFT $(\mathcal{A}, \hat{G}, \dots)$ has a

$$\text{Witten Index} = \chi = \mathrm{tr}_{\mathcal{A}_R}\left((-1)^F q^{L_0 - \frac{c}{24}}\right) \in \mathbb{Z}.$$

Zhu Modularity $\Rightarrow$ $\chi \cdot \eta^{2c} \in MF_c$. $\Rightarrow$ $\chi = 0$ if $c \neq 12k$.

<u>Goal</u>: Engineer SCFTs of small nonzero index.

<u>Motivation</u>:

<u>conjecture</u>:

"topological modular forms"

$$\begin{tikzcd}
 & TMF \arrow[dr] & \\
\{SQFTs\} \arrow[ur] \arrow[rr, "\text{``elliptic index''}", "\chi(q)\,\eta(q)^{2c}"'] & & MF \\
\cup & & \cup \\
\{SCFTs\} & & \\
\cup & & \\
\left\{\begin{matrix}\text{holomorphic} \\ \text{SCFTs}\end{matrix}\right\} \arrow[rr, "\chi \cdot \Delta^k"'] & & \mathbb{Z}[\Delta]
\end{tikzcd}$$

This is <u>the</u> most important deformation invariant of SQFTs.

TMF $\to$ MF is <u>not surjective</u>.

want to test the conj.

How to engineer a holomorphic CFT?

Naive strategy: Grab any VOA $\mathcal{B}$. Try to expand it to a holomorphic one.

If $\mathcal{A} \supseteq \mathcal{B}$, then $\mathcal{A} = \mathcal{B} \oplus \bigoplus_i M_i$ ← some $\mathcal{B}$-modules.

$\mathrm{Rep}(\mathcal{A}) < \mathrm{Rep}(\mathcal{B})$.

This strategy almost always fails. Most VOAs cannot be expanded to holomorphic ones.

E.g.: Will never work if $c \notin 8\mathbb{Z}$.

In general, whether $\mathcal{B}$ is expandable is a question about the structure of $\mathrm{Rep}(\mathcal{B})$.

Example: Lattice CFTs:

Start with $\mathbb{B} = Bos(n)$, the VOA of n free bosons.

Physically, these are the chiral operators in a non-holomorphic CFT of n indep. harmonic functions.

$Rep(\mathbb{B}) \cong \mathbb{R}^n$ with standard Euclidean metric.

$\mathbb{B}$ itself $\leftrightarrow M_0$ $\quad 0 \in \mathbb{R}^n$ origin.

Fact: $A = \bigoplus_{\ell \in L} M_\ell$ has a (automatically unique) (super) VOA str

i.e. $L \subseteq \mathbb{R}^n$ is a subgp, $\ell \cdot \ell' \in \mathbb{Z}$ $\forall \ell, \ell' \in L$.

iff $L \subseteq \mathbb{R}^n$ is an even (odd) lattice.

$\ell^2 \in 2\mathbb{Z}$; ℓ^2 sometimes odd.

$\{v \in \mathbb{R}^n \text{ s.t. } v \cdot \ell \in \mathbb{Z} \ \forall \ell \in L\}$

$Mod(A) \simeq L^* / L$. Holomorphic $\iff$ L is self-dual.

How to engineer SCFTs?

Naive strategy: start w/ $\hat{G} \in \mathcal{B}$ ↙ now a super VOA $= \mathcal{B}_0 \oplus \mathcal{B}_1$.

Classical example: $\mathcal{B} = \mathrm{Bos}(n) \times \mathrm{Fer}(n)$, susy that pairs them

"n free bosons, n free fermions"

Problem: If ψ is a free boson in your CFT, then ψ_0 is an odd operator on A_R and $\psi_0^2 = 1$. So $\chi = 0$.

True for any $\psi \in (A_{NS})_{1/2}$

Quantum example: Lattice VOA for $\sqrt{3}\,\mathbb{Z} \subseteq \mathbb{R}^1$ has susy.

In a lattice VOA, $(A_{NS})_{n/2}$ is gen. by operators "Γ_ℓ" for $\ell \in L$ with $\frac{\ell^2}{2} = \frac{n}{2}$. Susy $\in (A_{NS})_{3/2}$. $\frac{(\sqrt{3})^2}{2} = \frac{3}{2}$!

$\mathcal{B} = V(\sqrt{3}\,\mathbb{Z})$ has a susy, but it is not expandable,

because $\mathrm{Rep}(\mathcal{B}) = \dfrac{(\sqrt{3}\,\mathbb{Z})^*}{\sqrt{3}\,\mathbb{Z}} = \dfrac{\frac{1}{\sqrt{3}}\mathbb{Z}}{\sqrt{3}\,\mathbb{Z}} \simeq \mathbb{F}_3$

has no subgroups.

To expand $V(L)$ requires that $\mathrm{Rep}(V(L)) = L^*/L$ has an isotropic subgroup $C \subseteq L^*/L$.

Then

$$\begin{tikzcd} L+C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^* \arrow[d] \\ C \arrow[r, hook] & L^*/L \end{tikzcd}$$

is another lattice.

Isotropic: $C^\perp \supseteq C$.

pairing on $\mathbb{R}^n$ $\Rightarrow$ $U(1)$-valued pairing on L^*/L.

If you want $V(L+C)$ to be holomorphic, you need $C \hookrightarrow L^*/L$ to be self-dual: $C^\perp = C$.

Since $V(\sqrt{3}\mathbb{Z})$ has a susy, so does

$$V(\sqrt{3}\mathbb{Z})^n = V(\sqrt{3}\mathbb{Z}^n).$$

$L^*/L \simeq \mathbb{F}_3^n$. So:

$\boxed{\text{Every self-dual } C \subseteq \mathbb{F}_3^n \text{ gives a holomorphic SCFT.}}$

What is its index? $V(L)_R$ is gen by vectors in $L+w$, where $w \in \mathbb{R}^n$ is s.t. $w \cdot \ell = \ell^2 \bmod 2$ $\forall \ell \in L$.

Index = signed count of ground states.

= Index(C) from the beginning.

Last step:

The TMF conjecture predicts that the minimal index of a holomorphic SCFT of central charge $c = 12k$ is $\frac{24}{\gcd(k,24)}$. Self-dual code $\Rightarrow A$ w/ index 24.

To get smaller: If C has action by $\mathbb{Z}/k$, can pass to $B = A^{\mathbb{Z}/k}$ fixed sub-VOA, and try to expand in a different way to $A' = B + \ldots$

$$\mathrm{Index}(A') = \frac{\mathrm{Index}(A)}{k} + \text{contributions from } (\ldots).$$

If action has few enough fixed points, then $(\ldots)$ will be of high energy, so won't contribute to the index (= # ground states)