Higher Algebraic Closure

The Ohio State University, 27 April 2023

Theo Johnson-Freyd, Dalhousie & Perimeter

Based on jt wk in progress w/ David Reutter

These slides: categorified.net/OSU-Colloquium.pdf

more technically-detailed version of talk:
categorified.net/Feza.pdf

Why fermions?

Consider a quantum physical system with some species of "particles". Maybe they are quarks and leptons, or maybe they are localized excitations (Cooper pairs, phonons, ...) in some material. They can move around, and collide to form new particles.

Question: Can you assign, to each particle species X, Y, ..., a ~~Hilbert~~ Vector space $\mathcal{H}(X)$, $\mathcal{H}(Y)$, ... of "internal states"?

Requirements:

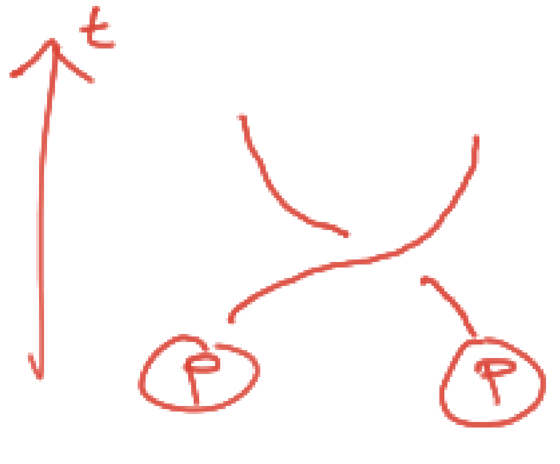
- System w/ particle X and particle Y $\mapsto \mathcal{H}(X) \otimes \mathcal{H}(Y)$.
- Superposition of X or Y $\longmapsto \mathcal{H}(X) \oplus \mathcal{H}(Y)$
- All physical processes are represented by ~~unitaries~~ isomorphisms.

In other words: consider the category of particle configurations. Does this category admit a nice functor to ~~Hilb~~ Vec?

Why fermions?

Answer: No. **E.g.:** Take two protons and prepare them in the same chosen state. Total system should be in state

$$|\text{chosen state}\rangle \otimes |\text{chosen state}\rangle \in \mathcal{H}(\text{proton}) \otimes \mathcal{H}(\text{proton}).$$



Now run the process that switches their locations. Should take $|\text{chosen}\rangle \otimes |\text{chosen}\rangle \mapsto |\text{chosen}\rangle \otimes |\text{chosen}\rangle$, and so if you allow both $\times$ and $||$, the superposition of these processes should interfere constructively.

But in fact, they interfere destructively.

In other words, must have $\times = -\,||$ for pairs of particles in identical states.

Why fermions?

Answer: No. **Cheap fix [Dirac?]:** **Super Vector spaces.**

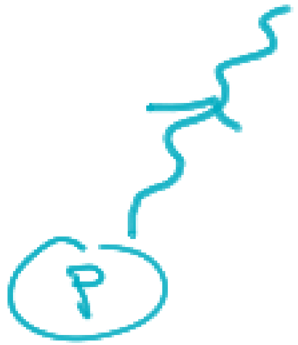
Defn: A super vector space is a $\mathbb{Z}/2$-graded vector space $V_{\bar{0}} \oplus V_{\bar{1}}$. Tensor product adds gradings (mod 2). Only difference: if you want to compare $v_{\bar{a}} \otimes w_{\bar{b}} \in V_{\bar{a}} \otimes W_{\bar{b}} \subseteq (V_{\bar{0}} \oplus V_{\bar{1}}) \otimes (W_{\bar{0}} \otimes W_{\bar{1}})$ to $w_{\bar{b}} \otimes v_{\bar{a}}$, you compare them with a factor of $(-1)^{\bar{a} \cdot \bar{b}}$.

In other words, you change the meaning of commutator.

Theorem [Deligne]: The cheap fix suffices*. Specifically, if $\mathcal{A}$ is any symmetric monoidal category which is not too large, and if $\mathcal{A} \neq 0$, then $\exists$ a sym mon functor $\mathcal{A} \to \mathrm{sVec}$.

* in at least 3+1 dimensions, so that

Why $\sqrt{-1}$?



From a relativistic perspective, particles are 1-dimensional because they trace a path as they move. Every particle gives a 1D observable that measures the expected work accrued by a given path.

There are also 0D observables — instantons — that measure values of fields at moments in spacetime.

An observable is an integral of motion if its value is locally constant in space and time.

Integrals of motion form a commutative algebra*:

* in at least 1+1 dimension

Why $\sqrt{-1}$?

Question: Can you assign, to each formally-real 0D observable, a real number so that multiplication of observables $\longleftrightarrow$ multiplication of numbers?

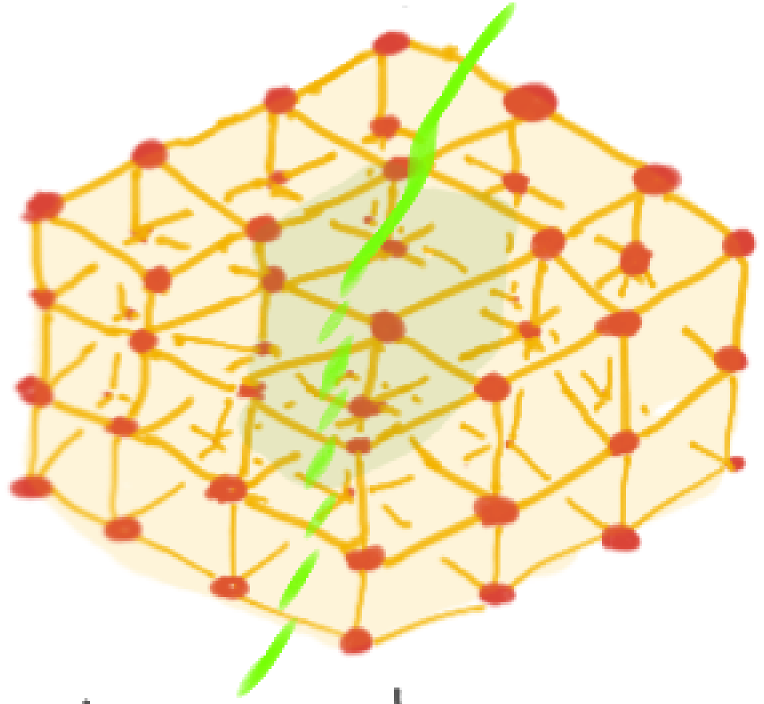
Answer: No. **E.g.:** There are superconducting materials which spontaneously generate magnetic fields. This breaks time-reversal symmetry. The ultraviolet T-symmetry instead becomes a formally-real 0D observable J s.t. $J^2 = -1$.

Cheap fix [Cardano?]: A ~~supernumber~~ complex number is a $\mathbb{Z}/2$-graded number $a_{\bar{0}} \oplus a_{\bar{1}}$. Multiplication adds gradings, but with an extra sign: $(a_{\bar{0}} \oplus a_{\bar{1}}) \cdot (b_{\bar{0}} \oplus b_{\bar{1}}) = \sum_{\bar{i},\bar{j} \in \mathbb{Z}/2} (-1)^{\bar{i}\bar{j}} a_{\bar{i}} b_{\bar{j}}$.

Theorem [Hilbert]: The cheap fix suffices: for any commutative $\mathbb{R}$-algebra $A \neq 0$ which is not too large, $\exists$ homomorphism $A \to \mathbb{C}$.

Going higher

In addition to instantons and particles, quantum systems can have extended objects. E.g.: Imagine a crystalization process in 3+1D of a chemical that likes to form a cubic lattice. If it crystalizes from the outside in, it might get stuck with defects where the crystal is off by one as it goes around.



Although this costs energy, the system cannot transition into a better configuration w/o a massive, energy-expensive, change all the way to ∞: the defect is topologically protected. On the other hand, the system will try to straighten out bends in the defect , so the defect behaves dynamically like a vibrating string. Even higher dimension: (mem)branes.

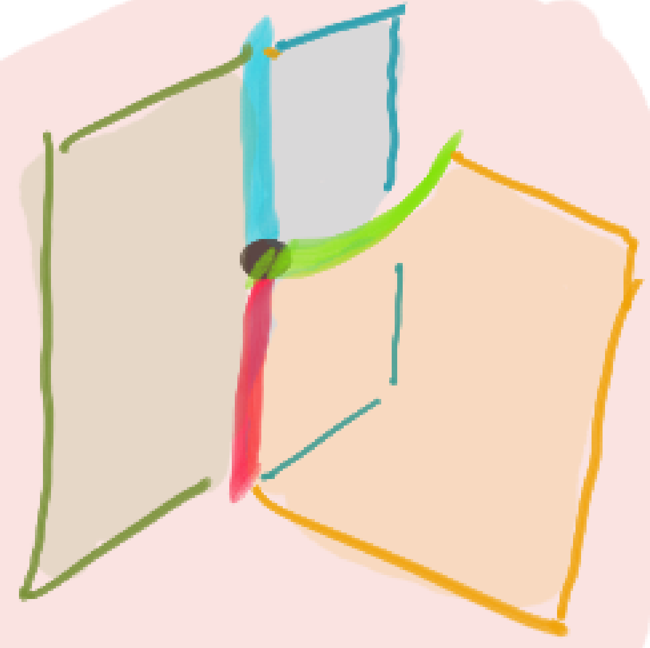
Going higher

Instanton = 0D.
particle = 1D
string = 2D
...

The correct language to describe the fusion and statistics of extend objects is higher category thy.

nD objects live in an n-category:

1-morphisms are (n-1)D junctions between objects.
2-morphisms are junctions between junctions.
Etc.



Main Question: Is there an nD generalization of $\mathbb{C}$, sVec, ...? I.e. an n-category $\mho^n$ s.t. every not too large non-zero commutative* n-cat maps to it?

Main Theorem [JF-Reutter]: If "n-category" ≡ "semisimple n-category," then YES: there is an infinite tower $\mho^0 = \mathbb{C}$, $\mho^1 = \text{sVec}$, $\mho^2$, $\mho^3$, ...

* aka symmetric monoidal. nD objects in at least 2n+2 D.

How to build $\mathcal{W}^\bullet$?

"not too large": this extension is reasonable.

Just like $\mathrm{sVec} \supseteq \mathrm{Vec} = \mathrm{Mod}(\mathbb{C})$, $\mathcal{W}^n \supseteq \mathrm{Mod}(\mathcal{W}^{n-1})$.

Want $\mathcal{W}^n$ nullstellensätzian aka algebraically closed.

So we are looking the algebraic closure of $\mathcal{A}^n := \mathrm{Mod}(\mathcal{W}^{n-1})$.

How to build algebraic closures? Look for Galois extensions.

How to build Galois extensions? Representation theory:

$$\left\{\begin{array}{l}\text{Galois extensions of } \mathbb{K} \\ \text{with Galois gp } G\end{array}\right\}_{/\text{iso}} \longleftrightarrow \left\{\text{surjections } \mathrm{Gal}^{abs}(\mathbb{K}) \xrightarrow{f} G\right\}_{/\text{conj.}}$$

$$\wr\!\!\cap$$

$$\left\{\begin{array}{l}\text{Sym mon functors} \\ \mathrm{Rep}_{\mathbb{K}} G \xrightarrow{F} \mathrm{Vec}_{\mathbb{K}}\end{array}\right\}_{/\text{iso}} \longleftrightarrow \left\{\text{maps } \mathrm{Gal}^{abs}(\mathbb{K}) \xrightarrow{f} G\right\}_{/\text{conj.}}$$

Specifically, given F, look at the commutative algebra $F(\underbrace{\mathcal{O}(G)}_{\text{fns on } G})$.
Iff F corresponds to a surjection, then $F(\mathcal{O}(G))$ is a field, and $\mathbb{K} \hookrightarrow F(\mathcal{O}(G))$ is Galois w/ Galois gp G.

How to build $\mathcal{W}^\bullet$?

$\left\{ \begin{array}{l} \text{Galois extensions of } \mathbb{K} \\ \text{with Galois gp } G \end{array} \right\} / \text{iso}$

$\updownarrow$

$\left\{ \begin{array}{l} \text{Sym mon functors} \\ \mathrm{Rep}_{\mathbb{K}} G \xrightarrow{F} \mathrm{Vec}_{\mathbb{K}} \end{array} \right\} / \text{iso}$

The search for such functors becomes much simpler if G is abelian (and $\mathrm{char}\, \mathbb{K} \nmid \#G$, and $\mathbb{K} \ni \sqrt[\#G]{1}$) as then $\{\text{simples in } \mathrm{Rep}(G)\} = G^\vee = \hom(G, \mathbb{K}^\times)$, and

$$\{\text{maps } \mathrm{Rep}\, G \to \mathrm{Vec}_{\mathbb{K}}\} \longleftrightarrow \{\text{maps } G^\vee \to \mathrm{Vec}_{\mathbb{K}}\} \longleftrightarrow \{\text{maps } G^\vee \to \mathrm{Vec}_{\mathbb{K}}^\times\}.$$

where $\mathrm{Vec}_{\mathbb{K}}^\times =: \underline{\mathrm{Pic}}(\mathbb{K})$ is the Picard spectrum of $\mathbb{K}$, which since $\mathbb{K}$ is a field is $\Sigma H\mathbb{K}^\times$. So we've translated the problem into stable homotopy thy:

$$\left\{ \begin{array}{l} \text{abelian Galois} \\ \text{extensions of } \mathbb{K} \end{array} \right\} \longleftrightarrow \left\{ \text{spectrum maps } \underbrace{HG^\vee}_{\text{abelian}} \longrightarrow \underline{\mathrm{Pic}}(\mathbb{K}) \right\}.$$

$= \mathrm{Ext}^1(G^\vee, \mathbb{K}^\times)$ if $\mathbb{K}$ is a field.

The* universal such map is the universal abelian Galois extension.

*if it exists

How to build $\mathcal{W}^\bullet$?

$\left\{ \begin{array}{c} \text{abelian Galois} \\ \text{extensions of } \mathbb{K} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{spectrum maps} \\ HG \to \underline{\mathrm{Pic}}(\mathbb{K}) \end{array} \right\}$.
$\uparrow$ abelian

The same thing works in higher algebra:

$$\left\{ \begin{array}{c} \text{abelian extensions} \\ \text{of } \mathcal{A} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{spectrum maps} \\ \text{into } \underline{\mathrm{Pic}}(\mathcal{A}) \end{array} \right\}.$$

Problems: • A priori, there is not nec. a universal ab gp G w/ a map $HG \to \underline{\mathrm{Pic}}(\mathcal{A})$. Existence is a nontrivial stable homotopy question.
• This finds the universal abelian Galois extension, not the alg. closure.

Solution: Using algebraic closedness of $\mathcal{W}^{n-1}$ and using that $\mathbb{Z}$ has global homological dim 1, we showed that $\mathcal{A}^n = \mathrm{Mod}(\mathcal{W}^{n-1})$ does have a universal abelian extension $\mathcal{W}^n$, and that it is algebraically closed among semisimple n-categories.

The moral reason $\mathrm{Gal}^{abs}(\mathcal{A}^n)$ is abelian is because it is π_n of something.

Computing the higher Galois Group

To compute $\pi_n \, \mathrm{Gal}(\mathcal{W}^\bullet/\mathbb{R}) \cong \Omega^n \mathrm{Gal}(\mathcal{W}^n/\mathcal{W}^{n-1})$, need to study $\underline{\mathrm{Pic}}(\mathrm{Mod}(\mathcal{W}^{n-1}))$. This is the space of

nD invertible phases with a semisimple top. boundary condition.

Specifically, need to understand kernel and cokernel of the map that forgets the boundary condition.

Turns out: There is a way to surger the boundary condition to make it simpler, analogous to surgery of manifolds.

The resulting surgery description of $\mathrm{Gal}(\mathcal{W}^\bullet/\mathbb{R})$ is very similar to the surgery description of PL.

Specifically, Gal looks like a profinite version of PL.

Computing the higher Galois Group

Sample Physical Corollary: A framed* QFT w/ nontrivial gravitational anomaly is necessarily gapless in the IR, unless the anomaly is an Arf-Kervaire invariant.

Equivalent statement: A nontrivial invertible phase of framed* matter necessarily has conducting edge modes, unless the phase is an Arf-Kervaire invariant.

Note [Hill-Hopkins-Ravenel]: The nD Arf-Kervaire invariant is framed-trivial unless $n = 2, 6, 14, 30, 62,$ and maybe 126.

* No Lorentz invariance. Equivalences can break Lorentz invariance.

Computing the higher Galois Group

Surgery methods, both ours and classically, break a bit in dimension ≈ 3, because of nonabelian knots and braids in $\mathbb{R}^3$.

In our case, it leads to a copy of the $\mathbb{C}^\times$-dual of the quantum Witt gp in $\pi_3 \mathrm{Gal}(\mathcal{W}^\circ/\mathbb{R})$.

Sample physical corollary [JF-Yu]: If you insist on working w/ bosonic TFTs, you see infinitely many invertible 6D phases.

qWitt introduces a weirdness that I cannot explain: it contains $\mathbb{Z}^{\oplus\infty}$ as a summand. So $\mathrm{Gal}(\mathcal{W}^\circ/\mathbb{R})$ is not profinite.

Credit where it's due

- Heavily inspired by Hopkins, who supplied many computation hints, and also who first asked for a universal tower $\mathcal{W}^\bullet$, and who analyzed $\mathcal{W}^\times$.
- Construction of $\mathcal{W}^3$ first done by Freed–Scheimbauer–Teleman. They told us their answer; we back-solved and generalized.
- Surgery for TFTs inspired by Lan–Kong–Wen, who introduced the 4D version.

THANKS