The Classification of Topological Orders

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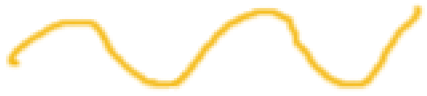
These slides available at http://categorified.net/OSU.pdf

Phases of matter

Microscopically, a system of matter might be described by

"The UV"

- molecules bumbing into each other 
- spins at each site in a lattice ↑↓↓↑↑↑↓↑↓
- quantum fields 
- ...

The phase of the system is its long-range effective macroscopic description.

"The IR"

Solid or fluid? Ferromagnetic? Conductive?

Phases of matter

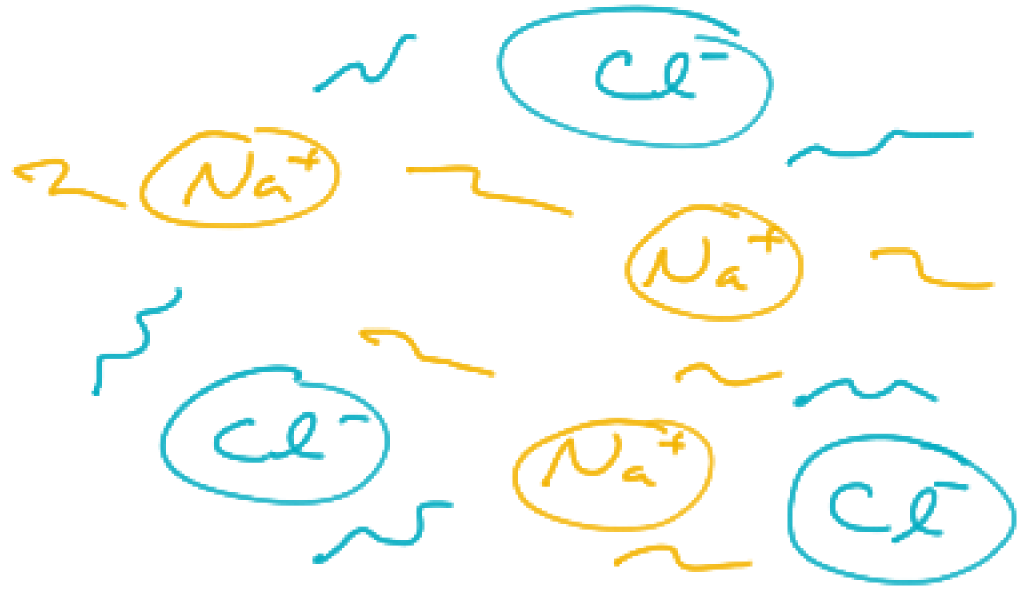
A humongous amount of physics research focuses on two overlapping questions:

* Classify phases of matter
* Understand the correspondence between UV and IR descriptions.

↳ In good situations, small changes to UV parameters do not affect the IR description. So the UV-to-IR map is many-to-one.

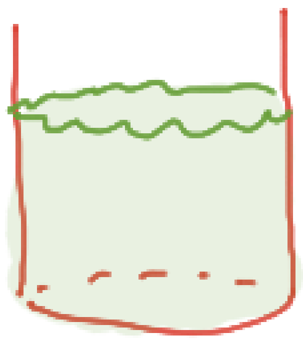
↳ The UV-to-IR map is not directly computable with perturbation theory, the usual tool in QFT.

Landau Paradigm



Consider anhydrous NaCl. Microscopically, this is a system of ions bumping into each other.

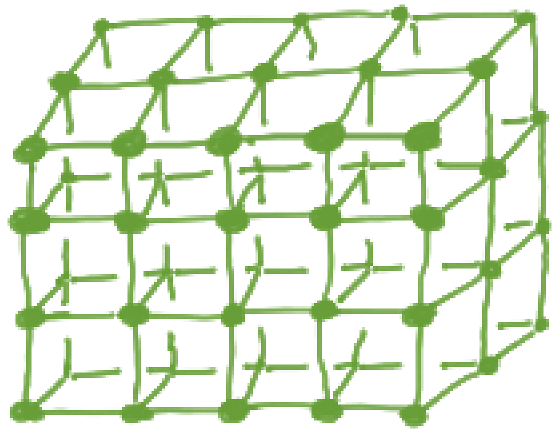
At high temperatures, it's macroscopically a liquid.



In particular, its phase is symmetric under all of $O(3) \ltimes \mathbb{R}^3$.

rotations + reflections translations

At low temperatures, it's macroscopically a crystal.



This is still effectively translation-invariant.

But the $O(3)$ symmetry spontaneously breaks to $\mathrm{Aut}(\square) \cong \underbrace{C_2 \times S_4}$

↳ measure w/ opacity, direction of cracks, etc.

same finite gp of order 48.

The crystal phase is ordered.

Landau Paradigm

The Landau Paradigm says that phases are classified by their patterns of symmetry and symmetry breaking.

It is remarkably effective, able to explain

- solid / liquid / gas
- (anti) ferromagnetism
- conductor / insulator
- relative strength of electro/weak forces

It suggests a solution to the UV-to-IR question:

(1) Identify all symmetries of the UV description. Work out basic properties like whether there is an anomaly: is the action on the Hilbert space linear or projective?

(2) Identify all possible patterns of symmetry breaking. Describe the phase with that order.

Topologically Ordered Phases

Phases that violate the Landau Paradigm emerged around the turn of the century. These phases are deeply quantum, with long range entanglement.

"Ordered by topology rather than symmetry"

useful for quantum computing!

Indeed, all of the physical data is encoded in the long range entanglement, and so the phase is locally trivial. In particular, there are no nontriv local operators in the IR description.

The local UV operators are "gapped out." The energy penalty required is too high for IR physics.

The local physics is invariant under all diffeomorphisms: it is a topological liquid.

Physics defn: A liquid is topological when its stress-energy tensor is central, i.e. $\mathbb{C}|vac\rangle$.

Topologically Ordered Phases

How to tell that such a phase is nontrivial?
It has extended operators.

local operator: supported at a point

extended operator: supported on a line

extended operator: supported on a surface

Nontopological example: In pure Maxwell thy (i.e. $U(1)$ gauge theory) in d dimensions, there are test electrons which measure the holonomy of the photon field along a curve.
Physically, inserting this operator measures the response an electrically charged partical would feel if it underwent that path in spacetime.

In maxwell theory, these operators are $\oint$ local operator.

There are also test magnetrons (magnetic monopole), which are $(d-3)$-dim

These operators do not commute: they detect each other.

Topologically Ordered Phases

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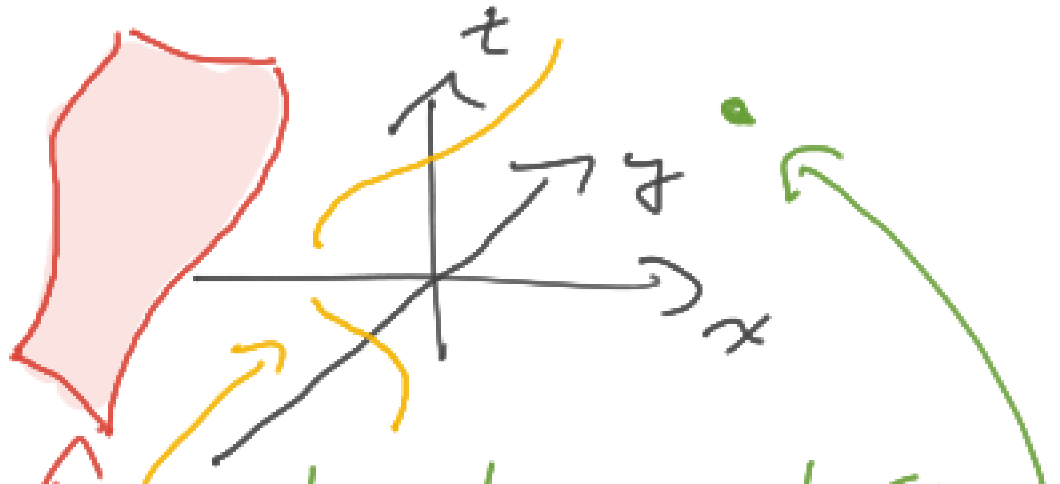


local operator: supported at a point

extended operator: supported on a line

extended operator: supported on a surface

Topological example: In the Toric Code ($\mathbb{Z}/2$ gauge thy) in d dimensions there are particle (aka line) test electrons and $(d-2)$-dimensional test magnetrons.

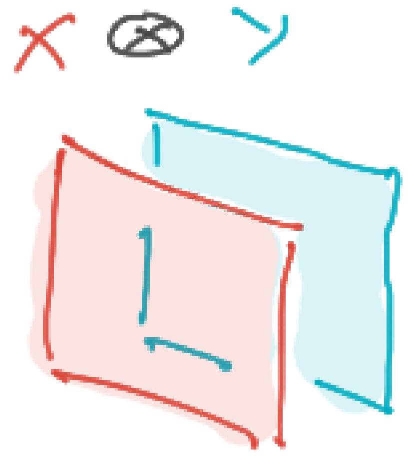
$d = 2+1$ commutation relation:

analogue of "$xy = -yx$"

analogues of "$x^2 = 1$".
Other coefficients are also allowed, e.g. in "twisted gauge theory".

Mathematical Axioms

A topological quantum liquid phase (aka a topological order) is determined by its "algebra" of extended operators. In $(n+1)$ dimensions, this "algebra" is a monoidal n-category $\mathcal{A}$.



- $\mathrm{Ob}(\mathcal{A})$ = n-dimensional operators

- $\mathrm{Ob}(\mathrm{End}_{\mathcal{A}}(\text{unit}))$ = $(n-1)$-dimensional operators
- $\mathrm{Ob}(\mathrm{End}_{\mathrm{End}_{\mathcal{A}}(\text{unit})}(\text{unit}))$ = $(n-2)$-dimensional operators

$\varphi \in \mathrm{End}_{\mathrm{End}(\text{unit})}(\text{unit})$

- Etc.

Mathematical Axioms

This monoidal n-category $\mathcal{A}$ should be

superpositions are allowed

Quantum: Linear, additive, and Karoubi complete

Topological: Very strong finiteness conditions

i.e. rigid finite semisimple, i.e. "multifusion"

forced by asking to be able to place phase on spacetimes like

i.e. central

Remotely detectable: The only invisible operators are scaler multiples of identity.

Robust: The only point operators are scalar multiples of identity

"fusion"

"Heisenberg uncertainty"

Otherwise, the system is in a critical state between two phases.

$\mathcal{A}$ is a higher categorical "central simple algebra".

Classification

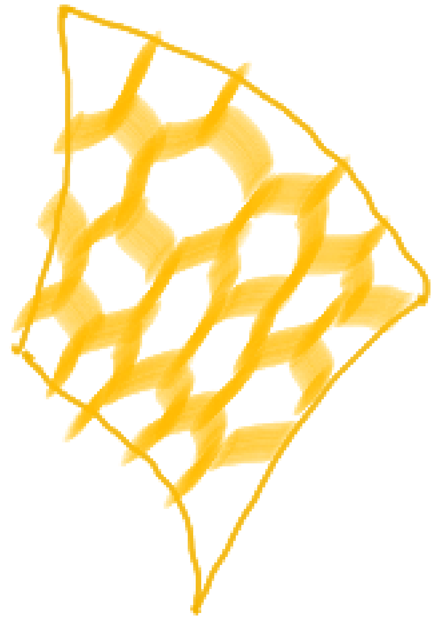
(0+1)D: $\{*\}$. Only point operators. Robustness $\Rightarrow$ all trivial

(1+1)D: $\{*\}$. Point and line operators.
Remote detectability $\Rightarrow$ line operators separate points
No points. So no lines.

In (0+1)D and (1+1)D, there is
no inherent topological order.

But symmetry-enriched systems are interesting because there can be anomalies.

(2+1)D: $\{MTCs\}$. Robust + remote detect $\Rightarrow$
[Wen]
all surface operators are networks of line operators



But the line operators are very rich.

- ∞ many.
- classification seems completely wild.

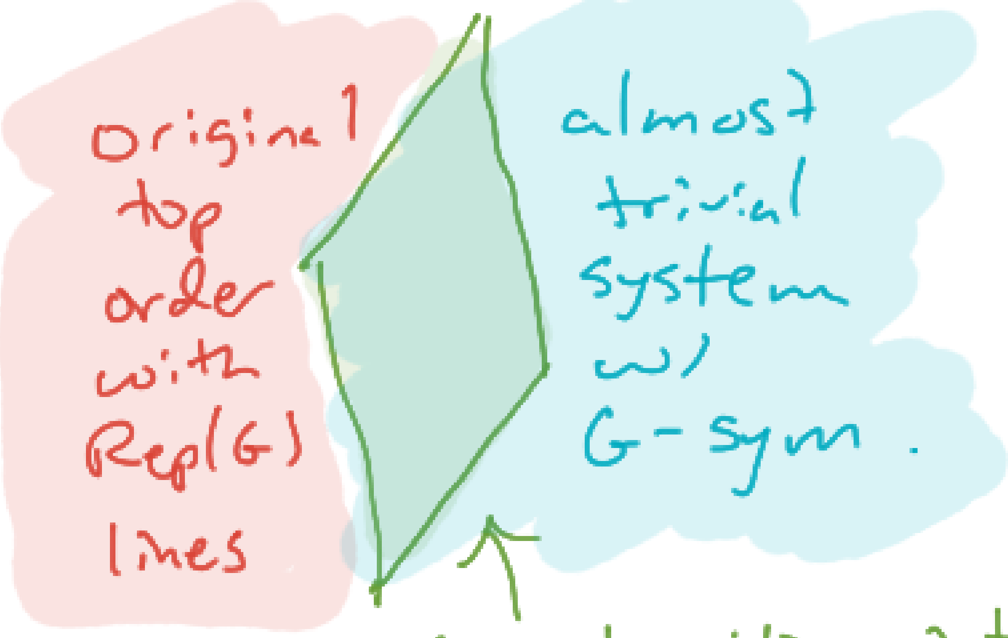
Classification

(3+1)D:

[Lan, Kong, Wen]
[Lan, Wen]
[JF]

- 3D operators are networks of 2D operators. So all data in the lines and surfaces.
- Line operators are symmetric monoidal.
  - ↳ so the bosonic lines are Rep(G) for some finite gp G.
  - ↳ can trigger a phase transition which "ungauges" a G-action.

"charged test bosons" in a G-gauge theory.



phase transition interface, a gapped interface.

<u>Thm</u>: • If no fermionic lines, then ordinary gauge thy classified by $H^4(G; U(1))$.

• If fermionic lines, then two choices for ungauged system, and $SH^4(G)$ choices for gauge thy action.

"super cohomology."

A certain generalized coh. thy.

<u>Thm [JF-Reutter]</u>: These two choices cannot be separated by a gapped interface.

Classification

(4+1)D:

[JF-Yu]

After coupling to a spin structure, all phases built by "Local fermion"

↳ gauging a "1-form symmetry"

↳ then gauging a "0-form symmetry"

The 0-form sym may act by electromagnetic duality on the 1-form gauge theory.

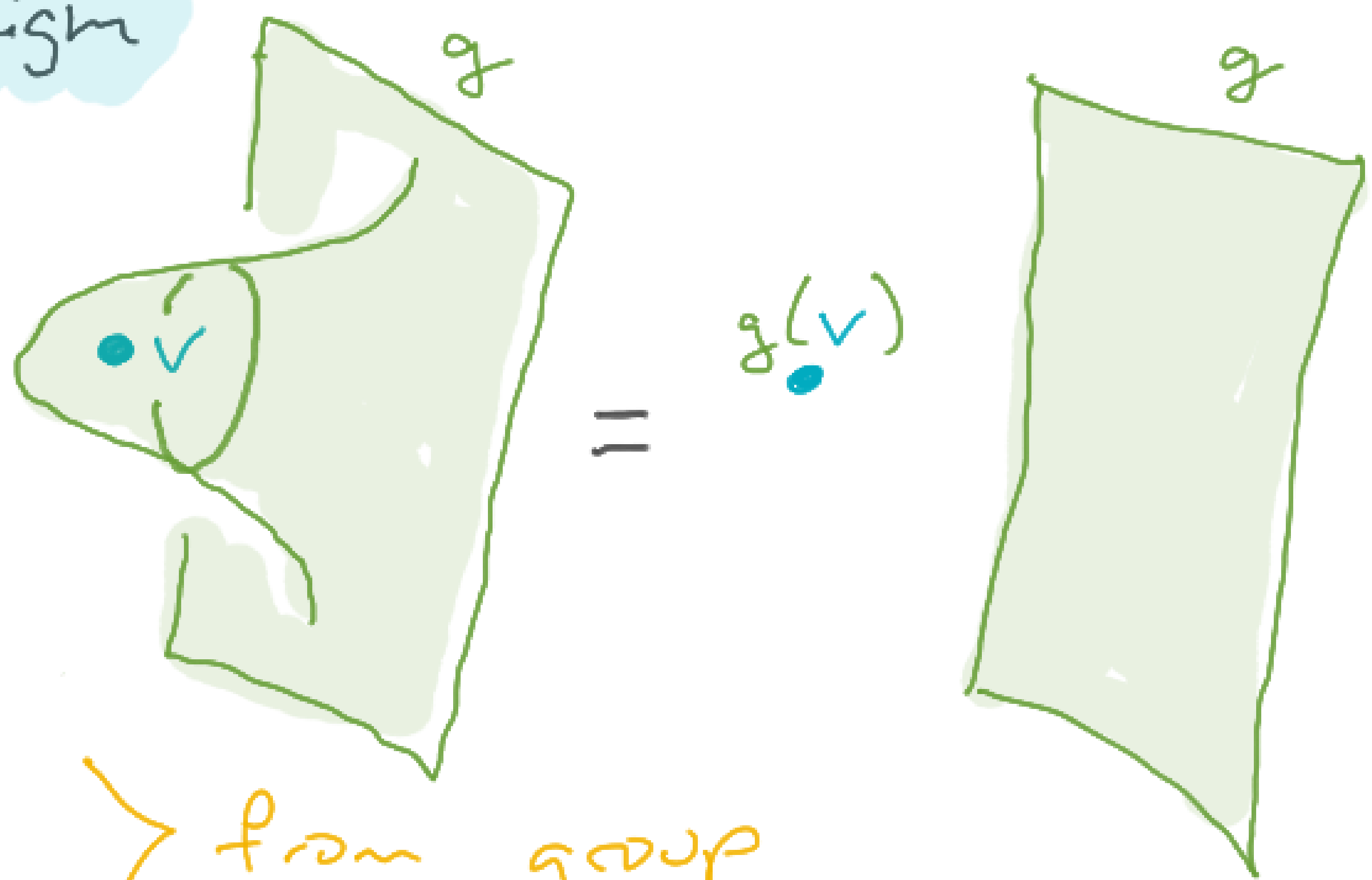
There is some manageable redundancy in this classification.

Before coupling to a spin structure, there are infinitely many phases which cannot be related by a gapped interface/phase transition.

Related to 2-torsion in the fermionic Witt group.

Higher Categorical Landau Paradigm

Symmetries can be encoded by codimension-1 symmetry defects. They are invertible and topological.



Higher homotopical symmetry $\leftrightsquigarrow$ higher codimensional sym. defects. } from group to higher group

Quantum superposition $\leftrightsquigarrow$ $g \cdot h = x \oplus y \oplus \cdots$ } from group to algebra.
$\leftrightsquigarrow$ non-invertible "symmetries"

Quantum field theory requires both generalizations. } higher algebra aka $\otimes$ n-cat!

"Categorical symmetry" restores the Landau Paradigm.